Singular Csck metric   Oct 29.

Alpha invariant for Kähler cone metrics.

Defn. Log Alpha invariant.

$\alpha_\beta := \sup \{ \alpha > 0 \mid \exists C$ s.t.

$$\sup_{\varphi \in H_\beta(\omega_\theta)} \int_X e^{-\alpha(\varphi - \sup_X \varphi)} \omega_\theta^n \le C \}.$$

- ω_θ is the reference metric.

Remark: In particular, if $\beta = 1$, $\alpha_\beta = \alpha$.

In the big classes. Ω.

Choose ω_{sr} be a C^∞, closed, $(1,1)$ form in Ω. In general, ω_{sr} can not be ≥ 0.

We can choose μ to be a positive measure on X, which puts no mass on pluripolar subsets.

Defn 4.12 arXiv:1803.09506

Defn (Alpha invariant for big class).

$$\alpha := \sup\left\{ \alpha > 0 \,\middle|\, \exists\, C \text{ s.t. } \sup_{\varphi \in \boxed{PSH(X, \hat{\omega}_{sr})}} \int_X e^{-\alpha(\varphi - \sup \varphi)} \boxed{d\mu} \le C \right\}.$$

- The definition is well-defined.
- The Alpha invariant for big class is continuous on the big cone.

Recall Properness $\Rightarrow$ Csck cone metrics.

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1. Criteria $\Rightarrow$ Properness.

Prop. Log J flow. Thm4.20 arXiv:1803.09506

We let ω_c be a Kähler cone metric.

1) $-\theta > 0$ 2) $[-C_\gamma \cdot \omega_c + (n-1)\theta] \wedge \omega_c^{n-2} > 0$

Then the log J flow for $\omega_\varphi = \omega_c + i\partial\bar\partial\varphi$,

$$\frac{\partial \varphi}{\partial t} = -C_\gamma + \mathrm{tr}_{\omega_\varphi}\theta - \gamma \frac{\omega_\theta^n}{\omega_\varphi^n}$$

converges.

Cor. $J^{\nu}_{-\theta}$ has lower bound.

$$J^{\nu}_{-\theta} = \underline{S}_\beta \cdot D(\varphi) + j_{-\theta}(\varphi) + \gamma \underline{J^{A}_{\omega_\theta}(\varphi)}.$$

prop 4.22 in arxiv: 1803.09506

Prop.$(*_c)$ Let $\Omega = [\omega_K]$ is a $\boxed{\text{Kähler}}$ class. big?

Assume that there is a const. η s.t.

$$(*_c)\begin{cases} 1) & 0 \le \eta < \frac{n+1}{n}\alpha_\beta, \\ 2) & C_1(X,D) < \eta\,\Omega, \\ 3) & \left(-\frac{C_1(X,D)\,\Omega^{n-1}}{\Omega^n} + \eta\right)\Omega > -(n-1)C_1(X,D) \end{cases}$$

$\Rightarrow$ The log K Energy is J^A-proper. Precisely:

$$\nu_\beta(\varphi) \ge \left(\frac{n+1}{n}\alpha_\beta - \eta\right) J^{A}_{\cancel{\#}\,(d_1)} - B, \quad \forall \varphi \in \mathcal{H}_\beta.$$

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prop 4.35 in arxiv: 1803.09506

Prop. Let Ω is big, ω_K is Kähler.

Suppose Ω satisfies $(*_c)$

$\Rightarrow$ There exists a sufficient small positive const. $\hat{t}$ s.t. $(*)_c$ also holds for $\Omega_t = \Omega + t[\omega_K]$ for any $0 < t \le \hat{t}$.

Prop 4.37 in arXiv: 1803.09506

Thm 1. Let Ω be a big and nef class on a Kähler manifold (X, ω_K).
Let Aut (M) to be trivial.
Suppose that Ω satisfies $(*_c)$.

$\Rightarrow$ Ω has the cscK Approximation property.

Precisely: $\Omega_t := \Omega + t[\omega_K]$, $0 < t \leq \hat{t}$,

1) ν_β is Proper in Ω_t

2) there exists cscK cone metric $\omega_t \in \Omega_t$.

3) ω_t has smooth approximation $\omega_{t,\varepsilon}$ in Ω_t.

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Section 5 in arXiv: 1803.09506

A priori estimates for ω_t / $\omega_{t,\varepsilon}$

$\Rightarrow$ $\omega_t \overset{?}{\Rightarrow} \omega_{sing}$ (singular cscK)

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$$\omega_t := \omega_K + i\partial\bar\partial \varphi_t .$$

$$\varphi_t \in PSH(\omega_K)$$

Singular CscK.

$$\varphi_t \longrightarrow \boxed{\varphi} \text{ in } L^1\text{-topology}.$$

$$\boxed{\|\varphi_t\|_\infty} \Rightarrow \varphi_t \to \varphi \text{ in } L^P\text{-topology}.$$

Thm. Prop 4.41 in arXiv: 1803.09506

Given the same assuption in Thm 1.

Assume $C_1(X, D) \geq 0$

the entropy $E_\beta^\varepsilon(\varphi_\varepsilon) = \frac{1}{V}\int_X F_\varepsilon e^{F_\varepsilon} \omega_{\theta_\varepsilon}^n$ is bdd.

$\Rightarrow$ • $\|\varphi_\varepsilon\|_\infty \leq C$.

• Partial 2nd estimate

$$\int_X (tr_{\omega_{b_\varepsilon}} \omega_{\varphi_\varepsilon})^p |S|_{h_E}^{\sigma} \omega_{b_\varepsilon}^n \leq C \quad \forall p \geq 1.$$

<u>Ques.</u> Ques 4.42 in arXiv: 1803.09506

$tr_{\omega_{b_\varepsilon}} \omega_{\varphi_\varepsilon} \leq C$? <u>Singular KE.</u>

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2. Log K Stability

Conj. (Log YTD conjecture)

The polarised pair $((X, L); D)$ admits a cscK cone metric

$\Leftrightarrow$ Log K-polystability.

$\Leftrightarrow$ Uniform log K stability.

Thm 4.1 in AHZ

Thm Suppose that $((X, L), D)$ admits a cscK cone metric of angle $2\pi\beta$.

Uniqueness Thm.

1. $((X, L), D)$ is log K-semistable.

Exience. Properness Thm.

2. $((X, L), D)$ is log K-polystable.
3. $((X, L), D)$ is G-uniformly log K-stable.

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3. The Csck cone Path

is defined to be the one parameter β family of Csck cone metrics of angle β.

- Log K energy.

Prop. Prop 2.12 and 2.13 in AHZ.

$$\nu_\beta = \nu_1 + (1-\beta)\frac{1}{V}\int_M \log|s|_h^2 (\omega_\varphi^n - \omega_0^n)$$

$$+(1-\beta) j_{\Theta_D}(\varphi) - (1-\beta)\underline{\frac{C_1(L_D)\Omega^{n-1}}{\Omega^n}} D_{\omega_0}(\varphi)$$

$$= \nu_1 + \underline{(1-\beta)}\left[D_{\omega_0, D} - \frac{Vol(D)}{V} D_{\omega_0}\right]$$

Cor. Thm 2.14 in AHZ.

The log K energy is linear in the cone angle β.

• Log Futaki invariant.

Defn. Let (X, L) be a polarised C^∞ projective variety.

We write $\mathrm{Aut}_0(X, L)$ to be the identity component of the group of biholomorphic automorphism on X whose action lifts to the total spae of L which is known to be a linear algebraic group.

$\mathfrak{aut}(X, L)$ for its Lie algebra.

Defn $((X, L), D)$

Let $D \subset X$ is smooth effective divisor.

$\mathrm{Aut}_0((X, L), D)$ to be subgroup of $\mathrm{Aut}_0(X, L)$ which preserves D.

Its Lie algebra $\mathrm{aut}((X,L)D)$ consists of holomorphic vector field on X which are tangential to D.

- v to be an element of $H^0(X, T_X)$.
- θ_v satisfies $i_v \omega = \sqrt{-1}\,\bar{\partial}\,\theta_v$.

Defn. Futaki invariant of $v \in H^0(X, T_X)$

$$\mathrm{Fut}(v) := -\int_X \theta_v (S(\omega) - \underline{S}_1) \frac{\omega^n}{n!}$$

Log Futaki invariant. cscK $\Rightarrow$ Fut $= 0$

$$\mathrm{Fut}_{D,\beta}(v) := \frac{1}{2\pi} \mathrm{Fut}(v)$$

$$+ (1-\beta)\left(\int_D \theta_v \frac{\omega^{n-1}}{(n-1)!} - \frac{\mathrm{Vol}(D)}{V} \int_X \theta_v \frac{\omega^n}{n!}\right)$$

$\boxed{\text{cscK cone} \Rightarrow \mathrm{Fut}_{D,\beta} = 0}$

Prop 2.20 in AHZ.

Prop. Suppose that there exists $v \in \mathrm{aut}((X, L), D)$ s.t.

$$F_D = \int_D \theta_v \frac{\omega^{n-1}}{(n-1)!} - \frac{\mathrm{Vol}(D)}{V} \int_X \theta_v \frac{\omega^n}{n!} \neq 0.$$

$\Rightarrow$ the cone angle $2\pi\beta$ of the cscK cone metric is given by

$$\beta = 1 + \mathrm{Fut}(v) \Big/ F_D.$$

Prop 2.21 in AHZ

Prop. Let A be an ample line bundle on X. Then there exists $m_1 \in \mathbb{N}$ depending on X, A s.t. for all $m \geq m_1$ we have $\mathrm{Aut}_0((X, L), D_m) = 0$ for a generic member D_m of the linear system $|mA|$.

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proof by using Prop. (*).

Thm:

- Small angle solution for the Csck cone path.

Prop. (*c) Thm 5.25 in AHZ.

- Openness of the Csck cone path.

proof: in Section 8 in arXiv:1803.09506
(openness of the KE cone path).

① log Properness Thm (Existence Thm).
or ② Implicite Function Thm. ($C^{4,\alpha,\beta}$)

Remark. YTD Conjecture. Closdness.

KE.
- KE cone path — $Ric \wedge \omega$.
- classical contining path — $Ric \geq c$ — Ric
- KRF — $\|Ric\|_{L^p}$. Riemannian Convergence Theory

Csck.
- csck cone path — $\boxed{\|S\|_\infty}$.
- continuity path (chen) — $\inf S$
- CF — $\|S\|_{L^2}$

4. Defn of log K stability

Defn. (Test configuration).

A test configuration $(\mathcal{X}, \mathcal{L})$ for a polarised Kähler manifold (X, L) is a scheme $\mathcal{X}$ with a flat projective morphism $\pi: \mathcal{X} \to \mathbb{C}$ s.t.

- $\mathbb{C}^*$ action acts on $\mathcal{X}$ in such a way: π is $\mathbb{C}^*$-equivariant.

  with a linearisation of the $\mathbb{C}^*$ on $\mathcal{L}$.

- $\pi^{-1}(1) = (X, L)$.

The central fibre $\pi^{-1}(0)$ over $0 \in \mathbb{C}$ is denoted by $\mathcal{X}_0$

A log TC. is a $(\mathcal{X}, \mathcal{L}, \mathcal{D})$ for (X, L, D) • Additionally require $\mathcal{D} \subset \mathcal{X}$ is obtained by complementing the $\mathbb{C}^*$-orbit of D in $\mathcal{X} \setminus \pi^{-1}(0)$ with the flat limit over $0 \in \mathbb{C}$.

Defn. (Log K stability)

Log K semistable. $DF \geq 0$ for any log TC.

Log K polystable, $DF = 0$

$\Rightarrow$ log TC is product.

Log K stable, $DF = 0$

$\Rightarrow$ log TC is trivial.

Defn (log Donaldson-Futaki invariant)

$$DF = \frac{2(a_1 b_0 - a_0 b_1)}{a_0} + (1-\beta)\frac{a_0 \hat{b}_0 - \tilde{a}_0 b_0}{a_0}.$$

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