Quantization and Chiral index

Si Li

YMSC, Tsinghua University

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in honor of Prof Yau for his 73th birthday
Motivation: QFT and Index Theory
Topological quantum mechanics (TQM)

TQM leads to a path integral on the loop space

$$\int_{\text{Map}(S^1,X)} e^{-S/\hbar} \xrightarrow{\hbar \to 0} \int_X \text{(curvatures)}$$

Topological nature implies the exact semi-classical limit $\hbar \to 0$, which localizes the path integral to constant loops.

- LHS = the analytic index expressed in physics
- RHS = the topological index.

This is the physics “derivation” of Atiyah-Singer Index Theorem.
Algebraic Index Theorem

Given a deformation quantization $A_{\hbar}(M) = (C^\infty(M)[[\hbar]], \star)$ on a symplectic manifold $(X, \omega)$, there exists a unique linear map

$$\text{Tr} : A_{\hbar}(M) \to \mathbb{C}((\hbar))$$

satisfying a normalization condition and the trace property

$$\text{Tr}(f \star g) = \text{Tr}(g \star f).$$

Then

$$\text{Tr}(1) = \int_M e^{\omega_{\hbar}/\hbar} \hat{A}(M).$$

This is the simplest version of algebraic index theorem which was first formulated by Fedosov and Nest-Tsygan as the algebraic analogue of Atiyah-Singer index theorem.
In [Grady-Li-L 2017, Gui-L-Xu 2020] A rigorous connection between the effective BV quantization for topological quantum mechanics and the algebraic index theorem via a geometric description of low energy effective theory.
Witten’s “Index Theorem” on loop space

Replace $S^1$ by an elliptic curve $E$. (Witten: index of Dirac operators on loop space).

2d Chiral analogue of algebraic index?
Mirror Duality between **Calabi-Yau** Geometries

\[
\begin{align*}
\text{symplectic geometry} \quad \text{(A-model)} & \quad \overset{\text{mirror symmetry}}{\longleftrightarrow} \quad \text{complex geometry} \quad \text{(B-model)} \\
\int \text{Map}(\Sigma_g, X) \quad \text{(A-model)} & \quad \overset{\text{"Fourier transform"}}{\rightarrow} \quad \int \text{Map}(\Sigma_g, X') \quad \text{(B-model)} \\
\int \text{Holomorphic maps}(\Sigma_g, X) & \quad \overset{\text{localize}}{\rightarrow} \quad \int \text{Constant maps}(\Sigma_g, X') \\
\downarrow & \quad \downarrow \\
\text{Gromov-Witten Theory} & \quad \text{Hodge theory}
\end{align*}
\]

The B-model can be viewed as a suitable mysterious way to

"count constant surfaces"  

related to the variation of Hodge structures and its quantization.
Observables and Factorization algebras
A QFT is usually described by a manifold $X$ and the data of fields

$$\text{Spacetime}: X \quad \implies \quad \text{Fields}: \mathcal{E} = \Gamma(X, E).$$

- $\mathcal{E}$ is the space (called fields) where we will do calculus $\int_{\mathcal{E}}$

  $$\langle \mathcal{O} \rangle := \int_{\mathcal{E}} \mathcal{O} e^{S/\hbar} \quad \text{"Path integral"}$$

- Topology of $X$ leads to new structures in $\infty$-dim geometry
One algebraic structure associated to the topology of $X$ is

$$\text{observables} = \text{functions on fields}$$

Given an open subset $U \subset X$, we can talk about

$$\text{Obs}(U) = \text{observables supported in } U$$

Example: $\delta$-function.
Observables form an algebraic structure as follows: given disjoint open subset $U_i$ contained in an open $V$: $\bigsqcup_i U_i \subset V$

we have a factorization product for observables

$$\bigotimes_i \text{Obs}(U_i) \to \text{Obs}(V).$$

- **Physics:** OPE (operator product expansion)
- **Mathematics:** factorization algebra.
  - **Origin:** Beilinson-Drinfeld in 2d CFT
  - **Costello-Gwilliam:** (perturbative renormalized) QFT.
Example: \( \text{dim } X = 1 \) (topological quantum mechanics)

QFT in \( \text{dim } = 1 \) is quantum mechanics.

In the topological case, for any contractible open \( U \), \( \text{Obs}(U) = A \).

The factorization product doesn’t depend on the location and size:

\[
A \otimes A \rightarrow A.
\]

We find an (homotopy) associative algebra.
Example: $\dim X = 2$ (chiral conformal field theory)

The factorization product of 2d chiral theory is holomorphic.

\[
\mathcal{O}_1(z)\mathcal{O}_2(w) \sim \sum_n \frac{\mathcal{O}_1(n)\mathcal{O}_2(w)}{(z-w)^{n+1}}
\]

which is the 2d analogue of “associative product”. We find \(\infty\)-many binary operations $\mathcal{O}_1(n) \cdot \mathcal{O}_2$.

In this case, observable algebra forms a vertex algebra.
An important class of quantities are correlation functions of observables. They capture “global” information of the theory.

- **Local correlation**

\[ \langle O_1(x_1) \cdots O_i(x_i) \cdots O_n(x_n) \rangle, \quad x_i \in X. \]

It is singular when points collide, hence a function on

\[ \text{Conf}_n(X) := \{ x_1, \cdots, x_n \in X \mid x_i \neq x_j \text{ for } i \neq j \}. \]

- **Many interesting non-local information is hidden in**

\[ \int_{\mathcal{Z} \subset \text{Conf}_n(X)} \langle O_1(x_1) \cdots O_i(x_i) \cdots O_n(x_n) \rangle \]

which might be divergent and require further renormalization.
Batalin-Vilkovisky (BV) Quantization formalism
Homological methods (such as BRST-BV) arises in physics as a general method to quantize theories with gauge symmetries.
**BV algebra**

A **Batalin-Vilkovisky (BV)** algebra is a pair \((\mathcal{A}, \Delta)\) where

- \(\mathcal{A}\) is a \(\mathbb{Z}\)-graded commutative associative unital algebra.
- \(\Delta : \mathcal{A} \to \mathcal{A}\) is a linear operator of degree 1 such that \(\Delta^2 = 0\).
- The **BV bracket** \(\{-, -\} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}\) by

\[
\{a, b\} := \Delta(ab) - (\Delta a)b - (-1)^{|a|}a\Delta b, \quad a, b \in \mathcal{A}.
\]

\(\{-, -\}\) satisfies a version of graded Leibnitz rule.

**Example (Polyvector fields)**

The space of smooth polyvector fields with a divergence operator

\[
(PV^\bullet(X) = \Gamma(X, \wedge^\bullet T_X), \quad \Delta = \text{divergence})
\]

is a BV algebra.
BV quantization formalism

Roughly speaking, BV quantization in QFT leads to

- Factorization algebra $\text{Obs}$ of observables.
- $(\mathcal{C}_\bullet(\text{Obs}), d)$: a chain complex via algebraic structures of $\text{Obs}$.
- A BV algebra $(\mathcal{A}, \Delta)$ with a BV-$\int$ map $\int_{\text{BV}}: \mathcal{A} \to \mathbb{C}$.
- A $\mathbb{C}[\hbar]$-linear map satisfying

$$\langle - \rangle : \mathcal{C}_\bullet(\text{Obs}) \to \mathcal{A}(\!(\hbar)\!)$$

satisfies quantum master equation (QME)

$$(d + \hbar \Delta)\langle - \rangle = 0. \quad \text{(QME)}$$

This means it is a chain map intertwining $d$ and $\hbar \Delta$.
- Partition function: $\text{Index} = \int_{\text{BV}} \langle 1 \rangle$. 

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Example: Topological Quantum Mechanics (TQM)
Local observables: Weyl algebra

\[ \text{Obs}_{1d} = \mathcal{W}_{2n} = (\mathbb{C}[p_i, q^i][\hbar], *) \]

\[ (C_\bullet(\text{Obs}_{1d}), b) = \text{the Hochschild chain complex.} \]

BV algebra \((A_{1d}, \Delta) = (\widehat{\Omega}_\bullet(\mathbb{R}^{2n}), \mathcal{L}_\Pi)\). Here \(\Pi = \text{Poisson tensor. In physics, this describes the geometry of zero modes.} \)
\[ \langle - \rangle_{1d} : C_\bullet(\mathcal{W}_{2n}) \rightarrow A_{1d}(\hbar) \text{ where} \]

\[ \langle O_0 \otimes O_1 \cdots \otimes O_m \rangle_{1d} \quad \text{where } O_i \in \mathcal{W}_{2n} \]

\[ = \int_{t_0 = 0 \prec t_1 \prec \cdots \prec t_m \prec 1} \left\langle O_0(\varphi(t_0))O_1^{(1)}(\varphi(t_1)) \cdots O_m^{(1)}(\varphi(t_m)) \right\rangle_{\text{free}} \]

It satisfies

\[ \text{QME} \quad (b + \hbar \Delta)\langle - \rangle_{1d} = 0 \]

Here \( b \) is the Hochschild differential.

Ref: \[ \text{Gui-L-Xu, 2020} \]
These data glues [Fedosov] to give a Weyl bundle $\mathcal{W}(X) \to X$.

- [Grady-Li-L]: BV quantum master equation is equivalent to Fedosov’s flat connection on $\mathcal{W}(X)$.

- $\langle - \rangle_{1d}$ leads to a trace map on deformation quantized algebra, as explicitly described by [Feigin-Felder-Shoikhet].

- [Grady-Li-L, Gui-L-Xu]: BV quantization of TQM gives

$$
\langle 1 \rangle = \int_X e^{\omega_1 / \hbar} \hat{A}(X).
$$
2d Chiral Conformal Field Theory
<table>
<thead>
<tr>
<th>1d TQM</th>
<th>2d Chiral CFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^1$</td>
<td>$\Sigma$</td>
</tr>
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<td>Associative algebra</td>
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</tr>
</tbody>
</table>

**Associative product**

\[ \mathcal{O}_1 \ast \mathcal{O}_2 \]

**Operator product expansion**

\[ \mathcal{O}_1(z)\mathcal{O}_2(w) \sim \sum_n \frac{\mathcal{O}_1(n)\mathcal{O}_2(w)}{(z-w)^{n+1}} \]
A chiral $\sigma$-model

$$\varphi : \Sigma \to X$$

will produce a bundle $\mathcal{V}(X)$ of chiral vertex operator algebras

$$\mathcal{V}(X) \downarrow \quad X$$

This is the chiral analogue of Weyl bundle in TQM.
Theorem (L)

The BV quantization of the 2d chiral model is equivalent to solving a flat connection on the vertex algebra bundle $\mathcal{V}(X)$

$$D = d + \frac{1}{\hbar} \left[ \oint \mathcal{L}, - \right], \quad D^2 = 0$$

where $\mathcal{L} \in \Omega^1(X, \mathcal{V}(X))$ and $\oint \mathcal{L}$ is the associated chiral vertex operator fiberwise.

- This is the chiral analogue of Fedosov connection.
- BRST reduction of chiral models falls into this setup

$$\oint \mathcal{L} = \text{BRST operator}.$$  

Ref: [L] Vertex algebras and quantum master equation.
Elliptic chiral homology

- In [Zhu, 1994], Zhu studied the space of genus 1 conformal block (the 0-th elliptic chiral homology) and establish the modular invariance for certain class of VOA.

- Beilinson and Drinfeld define the chiral homology for general algebraic curves using the Chevalley-Cousin complex.

- Recently, [Ekeren-Heluani, 2018, 2021]: an explicit complex expressing the 0th and 1st elliptic chiral homology.
Intuitively, the chiral differential in the chiral complex can be viewed as a 2d chiral analogue of the Hochschild differential $b$.

\[ d_{ch} = \sum \]

\[ b(a_0 \otimes \cdots \otimes a_p) = (-1)^p a_p a_0 \otimes \cdots \otimes a_{p-1} + \sum_{i=0}^{p-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p. \]
We briefly review the construction of Beilinson and Drinfeld.

- $\mathcal{M}(X)$: category of (right) $D$-modules on $X = \Sigma$
- $\mathcal{M}(X^S)$: category of (right) $D$-modules on $X^S$

$M \in \mathcal{M}(X^S)$ is a rule that assigns to each finite index set $I \in S$ a right $D$-module $M_{X^I}$ on $X^I$ (satisfying some compatibility conditions.)

- There is an exact fully faithful embedding

$$\Delta^{(S)}_*: \mathcal{M}(X) \hookrightarrow \mathcal{M}(X^S)$$

defined by $(\Delta^{(S)}_* M)_{X^I} := \Delta^{(I)}_* M$, where $\Delta^{(I)}: X \hookrightarrow X^I$.

The category $\mathcal{M}(X^S)$ carries a tensor structure $\otimes^{ch}$ and a chiral algebra $\mathcal{A}$ is a Lie algebra object via $\Delta^S_*$. 
We consider the **Chevalley-Eilenberg** complex

\[(C(\mathcal{A}), d_{CE}) = (\bigoplus \text{Sym}_{\text{ch}}^\bullet (\Delta^S_\ast \mathcal{A}[1]), d_{CE}),\]

which is a complex in \(\mathcal{M}(X^S)\).

The chiral homology (complex) \(\mathcal{C}^\text{ch}(X, \mathcal{A})\) is defined by \(R\Gamma_{DR}(X^S, C(\mathcal{A}))\), where

- **Dolbeault Resolution**
- **Chevalley-Eilenberg**
- **Spencer Resolution**
Example: $\beta\gamma - bc$ system

The VOA $\mathcal{V}^{\beta\gamma-bc}$ of $\beta\gamma - bc$ system is the chiral analogue of Weyl/Clifford algebra. It gives rise to a chiral algebra (in the sense of Beilinson and Drinfeld) $\mathcal{A}^{\beta\gamma-bc}$ on a Riemann surface $X = \Sigma$.

$$\beta(z)\gamma(w) \sim \frac{1}{z - w} \quad b(z)c(w) \sim \frac{1}{z - w}.$$  

The factorization homology (complex)

$$(\mathcal{C}_\bullet(\mathcal{V}^{\beta\gamma-bc}(\mathfrak{h})), d_{ch})$$

in the BV formalism will be the chiral chain complex $\mathcal{C}^{ch}(X, \mathcal{A}^{\beta\gamma-bc})$. 
Theorem (Gui-L)

Let $X$ be an elliptic curve $E_{\tau}$. We can construct an explicit map

$$\langle - \rangle_{2d} : C^{\text{ch}}(X, A^{\beta \gamma - bc}) \to A_{2d}((\hbar))$$

satisfying

$$QME : \quad (d_{\text{ch}} + \hbar \Delta) \langle - \rangle_{2d} = 0.$$ 

Roughly speaking, this map is defined by

$$\langle O_1 \otimes \cdots \otimes O_n \rangle_{2d} := \int_{E^n_{\tau}} \langle O_1(z_1) \cdots O_n(z_n) \rangle.$$

- $\langle O_1(z_1) \cdots O_n(z_n) \rangle$ is local correlation (via Feynman rules).
- $\int$ is the regularized integral introduced by [L-Zhou]. This is a geometric renormalization method for 2d chiral QFT.
- The BV trace map leads to Witten genus.
The issue of singular integral and renormalization

We need to understand the integral of local correlators

$$\int_{\Sigma^n} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle \equiv ?$$

Unlike the situation in topological field theory, $\langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle$ is very singular along diagonals and there is no way to extend it to certain compactification of $\text{Conf}_n(\Sigma)$. 
Regularized integral (L-Zhou 2020)

Let us first consider the integral of a 2-form $\omega$ on $\Sigma$ with meromorphic poles of arbitrary orders along a finite subset $D \subset \Sigma$. Locally we can write $\omega = \frac{\eta}{z^n}$ where $\eta$ is smooth 2-form and $n \in \mathbb{Z}$.

We can decompose $\omega$ into

$$\omega = \alpha + \partial \beta$$

where $\alpha$ is a 2-form with at most logarithmic pole along $D$, $\beta$ is a $(0,1)$-form with arbitrary order of poles along $D$, and $\partial = dz \frac{\partial}{\partial z}$ is the holomorphic de Rham. We define the regularized integral

$$\int_{\Sigma} \omega := \int_{\Sigma} \alpha + \int_{\partial \Sigma} \beta$$

This does not depend on the choice of the decomposition.
$f_\Sigma$ is invariant under conformal transformations. The conformal geometry of $\Sigma$ gives an intrinsic regularization of the integral $\int_\Sigma \omega$.

The regularized integral can be viewed as a “homological integration” by the holomorphic de Rham $\partial$

$$\int_\Sigma \partial(-) = \int_{\partial \Sigma} (-).$$

The $\bar{\partial}$-operator intertwines the residue

$$\int_\Sigma \bar{\partial}(-) = \text{Res}(-).$$
In general, we can define

\[
\int_{\Sigma^n} (-) := \int_{\Sigma} \int_{\Sigma} \cdots \int_{\Sigma} (-).
\]

This gives a **rigorous** and **intrinsic** definition of

\[
\langle O_1 \otimes \cdots \otimes O_n \rangle_{2d} := \int_{\Sigma^n} \langle O_1(z_1) \cdots O_n(z_n) \rangle.
\]

It exhibits all the required properties:

- **Holomorphic Anomaly Equation.** \textbf{(L-Zhou, in preparation)}
- **Contact equations.** \textbf{(Gui-L-Tang, in preparation)}
- ...
Elliptic chiral index (after Douglas-Dijkgraaf)

The partition function of a chiral deformation by a chiral lagrangian $\mathcal{L}$ is given by

$$\left\langle e^{\frac{1}{\hbar}} \int_{\Sigma} \mathcal{L} \right\rangle_{2d}.$$

If we quantize the theory on elliptic curve $\Sigma = E_\tau$,

$$\lim_{\bar{\tau} \to \infty} \left\langle e^{\frac{1}{\hbar}} \int_{E_\tau} \mathcal{L} \right\rangle_{2d} = \text{Tr}_{\mathcal{H}} q^{L_0 - \frac{c}{24}} e^{\frac{1}{\hbar} \oint dz \mathcal{L}}, \quad q = e^{2\pi i \tau},$$

where the operation $\lim_{\bar{\tau} \to 0}$ sends

almost holomorphic modular forms $\Rightarrow$ quasi-modular forms.

This can be viewed as an chiral algebraic index on the loop space. The regularized integral [L-Zhou] precisely explains $\bar{\tau} \to \infty$. 
Theorem (L-Zhou 2020)

Let $\Phi(z_1, \cdots, z_n; \tau)$ be a meromorphic elliptic function on $\mathbb{C}^n \times \mathbb{H}$ which is holomorphic away from diagonals. Let $A_1, \cdots, A_n$ be $n$ disjoint $A$-cycles on $E_\tau$. Then the regularized integral

$$\int_{E_\tau^n} \left( \prod_{i=1}^n \frac{d^2 z_i}{\text{im} \, \tau} \right) \Phi(z_1, \cdots, z_n; \tau)$$

lies in $\mathcal{O}_\mathbb{H}[\frac{1}{\text{im} \, \tau}]$ and

$$\lim_{\tau \to \infty} \int_{E_\tau^n} \left( \prod_{i=1}^n \frac{d^2 z_i}{\text{im} \, \tau} \right) \Phi(z_1, \cdots, z_n; \tau) = \frac{1}{n!} \sum_{\sigma \in S_n} \int_{A_1} dz_{\sigma(1)} \cdots \int_{A_n} dz_{\sigma(n)} \Phi(z_1, \cdots, z_n; \tau).$$

In particular, $\int_{E_\tau^n}$ gives a geometric modular completion for quasi-modular forms arising from $A$-cycle integrals.
## Algebraic Index Theory vs Elliptic Chiral Index Theory

<table>
<thead>
<tr>
<th>1d TQM</th>
<th>2d Chiral CFT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Associative algebra</td>
<td>Vertex operator algebra</td>
</tr>
<tr>
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<td>Chiral homology</td>
</tr>
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</tr>
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<td>$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}<em>n \rangle</em>{1d} =$ integrals on the compactified configuration spaces of $S^1$</td>
<td>$\langle \mathcal{O}_1 \otimes \cdots \otimes \mathcal{O}<em>n \rangle</em>{2d} =$ regularized integrals of singular forms on $\Sigma^n$</td>
</tr>
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<td>Elliptic Chiral Algebraic Index</td>
</tr>
</tbody>
</table>

Joint work with **Zhengping Gui**. arXiv:2112.14572 [math.QA]
Thank You and Happy Birthday Yau!