Symmetries of the Schrödinger flow on $\mathbb{C}P^n$

Chuu-Lian Terng[1]

[1]Department of Mathematics
University of California, Irvine

Chern: a great geometer of the 20th century
a conference for the 110th anniversary of the birth of
Professor Shing-Shen Chern
at Tsinghua University
October 10-14, 2021

We (Richard Palais and I) visited Chern at Nankai Institute in January 2003

Outline of my Talk

- Chern started me in soliton theory: Classical Bäcklund Theorem for surfaces in $\mathbb{R}^3$ with $K = -1$
- Joint work with K. Uhlenbeck on Schrödinger flow on $\mathbb{C}P^n$ (GNLS):
  - (i) The GNLS is gauge equivalent to Fordy-Kulish's Vector NLS.
  - (ii) Bäcklund transformations arise from the action of a rational loop group on the space of solutions.
  - (iii) Other symmetries: a Hamiltonian action of an infinite dimensional group, a Poisson action of a non-abelian loop group, and Virasoro action on tau functions.

Chern started me in soliton theory

- I went to UC Berkeley in 1976 as an instructor and Professor Chern was my advisor.
- In the summer of 1977, Chern gave a seminar on Bäcklund transformations (BTs) for the sine-Gordon equation

$$q_{xt} = \sin q, \qquad \text{SGE}$$

  which is the equation for surfaces in $\mathbb{R}^3$ with $K = -1$ and was one of the few soliton equations known at the time.
- He suggested to me and Keti Tenenblat to generalize Bäcklund's result to n-dimension and to investigate whether this gives a new soliton equation with n variables. We did have success in this project.

Surfaces in $\mathbb{R}^3$ with Gaussian curvature -1

- 1862 Edmond Bour showed that a surface M in $\mathbb{R}^3$ with $K \equiv -1$ admits local coordinates (x, t) such that f_x and f_t are unit asymptotic vectors and $\phi := \angle(f_x, f_t)$ satisfies the SGE $q_{xt} = \sin q$, where f is the immersion.

- Given a solution of the SGE, we can construct a surface in $\mathbb{R}^3$ with $K = -1$ unique up to rigid motion.

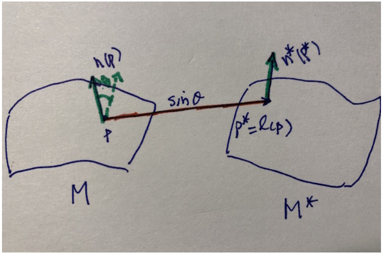
- The study of surfaces in $\mathbb{R}^3$ with $K \equiv -1$ is equivalent to the study of solutions of the SGE.

Bäcklund's Theorem (1883)

Let M, M^* be surfaces in $\mathbb{R}^3$, and $0 < \theta < \pi$ a constant. If $\ell : M \to M^*$ satisfies

1. $\overrightarrow{pp^*}$ is tangent to M and M^* at p and p^* resp.,
2. $||\overline{pp^*}|| = \sin\theta$,
3. the angle between the normal $n(p)$ to M and the normal $n^*(p^*)$ to M^* is θ,

for all $p \in M$ and $p^* = \ell(p)$. Then both M, M^* have $K \equiv -1$. (Such ℓ is called a Bäcklund Transformation (BT).)



Bäcklund Theorem Continued

- let $s = \tan\frac{\theta}{2}$, and q, q^* be the solutions of SGE given by M, M^*, then q, q^* satisfy

$$(BT)_{q,s} \qquad \begin{cases} q^*_x = q_x + 4s\sin\frac{q+q^*}{2}, \\ q^*_t = -q_t + \frac{2}{s}\sin\frac{q^*-q}{2}. \end{cases}$$

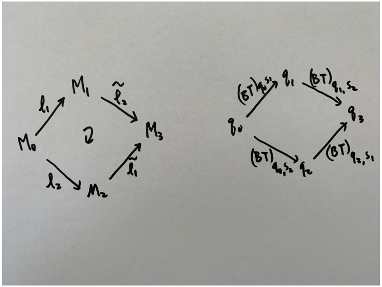
- If q is a solution of the SGE, then the system $(BT)_{q,s}$ is solvable for q^* and the solution q^* is a new solution of the SGE.

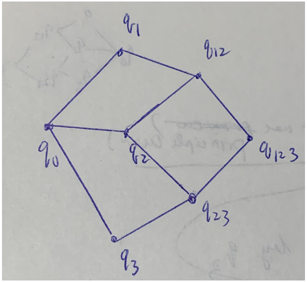
Bianchi Permutability for Bäcklund Transformations

Let $0 < \theta_1 \neq \theta_2 < \pi$, $s_i = \tan\frac{\theta_i}{2}$, and $\ell_i : M_0 \to M_i$ Bäcklund transformation with constant θ_i for $i = 1, 2$. Then there exist a unique surface M_3 and BTs $\tilde{\ell}_1 : M_2 \to M_3$ and $\tilde{\ell}_2 : M_1 \to M_3$ such that $\tilde{\ell}_1 \circ \ell_2 = \tilde{\ell}_2 \circ \ell_1$. Moreover, let q_0 be a solution of SGE, and q_i a solution of $(BT)_{q,s_i}$ for $i = 1, 2$, then q_{12} defined by

$$\tan\frac{q_{12} - q_0}{4} = \frac{s_1 + s_2}{s_1 - s_2} \tan\frac{q_1 - q_2}{4}$$

solves SGE, $(BT)_{q_1,s_2}$ and $(BT)_{q_2,s_1}$.





Soliton solutions of the SGE

- Apply BT to $q = 0$ to obtain a family of 1-soliton solutions

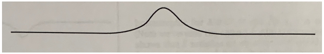
$$q^* = 4\tan^{-1}(e^{sx+s^{-1}t}) = 4\tan^{-1}(e^{\frac{s+s^{-1}}{2}X+\frac{s-s^{-1}}{2}T}),$$

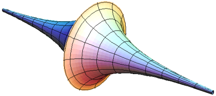
  where $s \in \mathbb{R} \setminus 0$ is a constant and (X, T) is the space-time coordinate $X = \frac{x+t}{2}$, $T = \frac{x-t}{2}$. This is a 1-soliton solution of the SGE. Note that $s > 1$ gives a kink, $0 < s < 1$ gives an anti-kink, and $s = 1$ gives a stationary solution.
- Apply Bianchi Permutability formula to 1-soliton solutions to obtain two parameter families of 2-soliton solutions of the SGE, ... etc.
- Apply Bianchi's formula with $s_1 = e^{i\theta}$ and $s_2 = -e^{-i\theta}$, we obtain a solution of the SGE that is periodic in the time variable T, which ia called a breather.

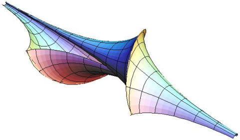
1- soliton solutions of SGE & corresponding $K = -1$ surfaces

If we apply BT to the trivial solution $q = 0$ with

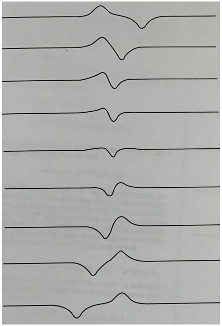
1. $s = 1$, we obtain a stationary solution and pseudo-sphere,
2. $s \neq 1$, we obtain the Dinni surfaces.

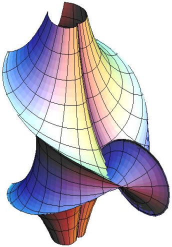




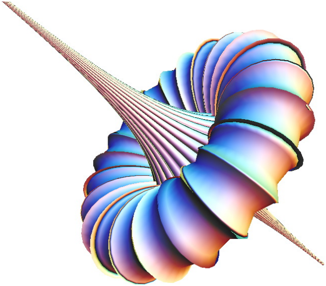


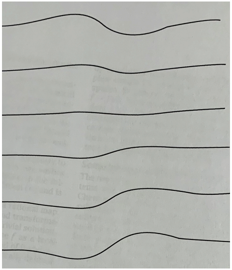
2-soliton solution of SGE $s_1 \neq s_2 \in \mathbb{R}$, Kuen surface





2-soliton Breather solution of SGE $s_1 = e^{i\theta} = -\bar{s}_2$





Schrödinger flow on $\mathbb{C}P^n$

- The Schrödinger flow on $\mathbb{C}P^n$ is the equation,

$$\gamma_t = J_\gamma(\nabla_{\gamma_x}\gamma_x),$$

  for $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{C}P^n$, where J is the complex structure and ∇ is the Levi-Civita connection on $\mathbb{C}P^n$.

- $\langle A_1, A_2 \rangle = -\mathrm{tr}(A_1 A_2)$ is an inner product on $u(n+1)$.

$$\{A \in u(n+1) \mid A = gag^{-1} \text{ for some } g \in U(n+1)\}$$

  with the induced metric from $u(n+1)$ is an isometric embedding of $\mathbb{C}P^n$ in $u(n+1)$ (the Cartan embedding), where $a = \mathrm{diag}(i, -i, \ldots, -i)\}$.

- Use the Cartan embedding, Schrödinger flow on $\mathbb{C}P^n$ is

$$\gamma_t = [\gamma, \gamma_{xx}]. \qquad \text{GNLS}$$

Moving frames and differential invariants for curves in $\mathbb{C}P^n$

Given smooth $\gamma : \mathbb{R} \to \mathbb{C}P^n$, then $\exists\ g : \mathbb{R} \to U(n+1)$ satisfying

$$\gamma(x) = gag^{-1}, \quad u := g^{-1}g_x = \begin{pmatrix} 0 & -q^* \\ q & 0 \end{pmatrix}$$

for some $q \in C^\infty(\mathbb{R}, \mathbb{C}^n)$.

- We call g a moving frame and u the differential invariant defined by g along γ.
- If g_1 is another moving frame along γ, then there is a constant $k \in U(n+1)_a = U(1) \times U(n)$ such that $g_1 = gk$. Note the the differential invariant defined by g_1 is $k^{-1}uk$.
- This is similar to the parallel normal frame along curves in $\mathbb{R}^3$ and its differential invariants, principal curvatures, depend on the choice of parallel normal frame.

Relation between GNLS and VNLS

Theorem (Terng-Uhlenbeck 06)
If γ is a solution of $\gamma_t = [\gamma, \gamma_{xx}]$, then there exists $g : \mathbb{R}^2 \to U := U(n+1)$ satisfying

(1) $\gamma = gag^{-1}$,

(2) $\begin{cases} g^{-1}g_x = u, \\ g^{-1}g_t = P_{-1}(u) := [a, u_x] - \frac{1}{2}[u, [a.u]] \end{cases}$,

for some $u = \begin{pmatrix} 0 & -q^* \\ q & 0 \end{pmatrix}$ with $q \in \mathbb{C}^{n\times 1}$.

Moreover,

- differential invariant u satisfies $u_t = i(u_{xx} + 2||u||^2 u)$, i.e.,

$$q_t = i(q_{xx} + 2||q||^2 q) \qquad \textit{VNLS}$$

- if g_1 also satisfies (1)-(2) above, then there is a constant $k \in \mathcal{U}_a = u(1) \times u(n)$ such that $g_1 = gk$.

Continue

- Conversely, if u is a solution of the VNLS and g is a solution of system (2), then $\gamma = gag^{-1}$ is a solution of the GNLS.
- u is a solution of the VNLS and c a constant in $U_a = U(1) \times U(n)$ then cuc^{-1} is also a solution of the VNLS.
- A solution γ of the GNLS maps to a U_a-orbit of solutions of VNLS.

Deriving Lax pair for VNLS from an action of $\mathbb{R}^*$

- γ satisfies the GNLS $\gamma_t = [\gamma, \gamma_{xx}]$ if and only if

$$\theta = \gamma \mathrm{d}x + (\gamma + [\gamma, \gamma_x])\mathrm{d}t \quad \text{is flat.}$$

- $\mathbb{R}^*$ action: Let γ be a solution of the GNLS and $\lambda \in \mathbb{R}^* = \mathbb{R} \setminus 0$, then $\tilde{\gamma}(x,t) = \gamma(\lambda^{-1}x, \lambda^{-2}t)$ is again a solution of the GNLS. Hence $\tilde{\theta} = \tilde{\gamma}\mathrm{d}x + (\tilde{\gamma} + [\tilde{\gamma}, \tilde{\gamma}_x])\mathrm{d}t$ is also flat. Let $\tilde{x} = \lambda^{-1}x, \tilde{t} = \lambda^{-2}t$. Then

$$\Theta = \lambda\gamma \mathrm{d}x + (\lambda^2\gamma + [\gamma, \gamma_{xx}])\mathrm{d}t$$

  is flat for all $\lambda \in \mathbb{R} \setminus 0$. This is the Lax pair for GNLS, one of the key properties of soliton equations.
- The gauge transform $g^{-1}\Theta g + g^{-1}\mathrm{d}g$ is

$$(a\lambda + u)\mathrm{d}x + (a\lambda^2 + u\lambda + ([a, u_x] - \frac{1}{2}[u, [a, u]]))\mathrm{d}t,$$

  which is a flat connection on the (x,t)-plane for all parameter $\lambda \in \mathbb{C}$. This is the Lax pair for VNLS.

How do we know a geometric PDE is a soliton equation?

- The geometric PDE can be written as the flatness of certain connection.
- There is an $\mathbb{R}^*$-action on the space of solutions of the PDE.
- Use the $\mathbb{R}^*$-action, we can associate to each solution a family of flat connections, the Lax pair.
- The shape of the Lax pair tells us what the loop group and loop group splitting will give the equation (I will explain next).

For example, many classes of submanifolds in space forms, curve flows in homogeneous space, harmonic maps from $\mathbb{R}^2$ to a compact Lie group or a symmetric space, space-time monopole equation, ... etc are soliton equations.

Soliton hierarchies from splitting of loop algebras

developed by Adler (1979), Ablowitz-Kaup-Segur-Newell (1974), Kupershmidt-Wilson (1981) Zakharov and Shabat (1974), Drinfeld-Sokolov (1984). Here I use T-U version:

L_+, L_- subgroups of a loop group L such that $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$ as linear subspaces. We call such a pair $(\mathcal{L}_+, \mathcal{L}_-)$ a splitting of $\mathcal{L}$. A vacuum sequence is a linearly independent commuting sequence $\{J_j | j \geq 1\}$ in $\mathcal{L}_+$ that is generated by $J_1 = J$. For simplicity, we assume in this talk

$$J = a\lambda, \quad J_j = a\lambda^j.$$

Set

$$V = [J, \mathcal{L}_-]_+.$$

- Given $u \in C^\infty(\mathbb{R}, V)$, $\exists\, M : \mathbb{R} \to L_-$ satisfying

$$M^{-1}(\partial_x + J)M = \partial_x + J + u.$$

M is a reduced wave function and $f = M(0)$ is a formal scattering data for u.

- $\exists$ a unique $P(u) : \mathbb{R} \to \mathcal{L}$ satisfying

$$[\partial_x + J + u, P(u)] = 0.$$

In fact, $P(u) = M^{-1} J M$.

- Write

$$P(u)(\lambda) = J + u + P_{-1}(u)\lambda^{-1} + P_{-2}(u)\lambda^{-2} + \cdots .$$

Then $P_j(u)$ is a polynomial differential of u,

- $$u_t = [\partial_x + u, P_{-(j-1)}(u)]$$

is an evolution PDE on $C^\infty(\mathbb{R}, V)$, call the *j*-th flow.

Poisson structure

Let F_1, F_2 be functionals on $C_s^\infty(\mathbb{R}, V)$. Then

$$\{F_1, F_2\}(u) = \int_{\mathbb{R}} \mathrm{tr}([\nabla F_1(u), a]\nabla F_2(u))\mathrm{d}x$$

is a Poisson structure on $C_s^\infty(\mathbb{R}, V)$. Moreover,

- the j-th flow is the Hamiltonian flow for

  $$H_j(u) = \frac{1}{j} \oint \mathrm{tr}(P_{-(j+1)}(u)a)\mathrm{d}x$$

  with respect to $\{\,,\}$,
- $\{H_i, H_j\} = 0$, in particular these flows commute.

Dressing and formal inverse scattering (T-U 1997)

- Given $f_- \in L_-$, $f_+ \in L_+$, we factor

$$f_-^{-1} f_+ = \tilde{f}_+ \tilde{f}_-^{-1}, \quad \tilde{f}_\pm \in L_\pm.$$

  Then

$$f_+ \sharp f_- = \tilde{f}_-$$

  defines an action L_+ on L_-. This is the dressing action.

- The formal inverse scattering transform is the map

$$\mathcal{F} : L_- \to C^\infty(\mathbb{R}, V)$$

  defined by

$$\mathcal{F}(f) = u_f, \quad u_f(x) := \mathrm{Res}_\lambda([a, e^{xJ} \sharp f])$$

- Given $u \in C^\infty(\mathbb{R}, V)$, let M be a reduced wave function for u and $f = M(0) \in L_-$. Then $u = u_f = \mathcal{F}(f)$.

- If $f \in L_-$, then $u_f(x,t) := \mathcal{F}(e^{J_j t} \sharp f)$ is a solution of the j-th flow. In fact, we factor

$$f^{-1}(\lambda) e^{Jx + J_j t} = E(x,t,\lambda) M^{-1}(x,t,\lambda),$$

with $E(x,t,\cdot) \in L_+$ and $M(x,t,\cdot) \in L_-$. Then

$$E^{-1} E_x = J + u, \quad E^{-1} E_t = (\lambda^{j-1} P(u,\lambda))_+,$$

and u is a solution of the j-th flow. We call E the frame of the solution u.

- The dressing action of the subgroup

$$\{f \in L_+ | f(\lambda) \in U_a\}$$

on L_- gives a Poisson action on $C^\infty(\mathbb{R}, V)$ under $\mathcal{F}$ and its flows commute with the j-th flow.

The action of L_- and BTs (TU 2000)

- Given $f, g \in L_-$, then

  $$g * u_f = u_{fg^{-1}}$$

  defines an action of L_- on the space of solutions of the j-th flow. In fact, if E is the frame of a solution u of the j-th flow and $g \in L_-$, we factor

  $$gE(x,t,\cdot) = \tilde{E}(x,t,\cdot)\tilde{g}(x,t)$$

  with $\tilde{E}(x,t,\cdot) \in L_+$ and $\tilde{g}(x,t) \in L_-$, then

  $$\tilde{u} = u + [a, \mathrm{Res}_\lambda\, \tilde{g}] = g * u$$

  is a new solution of the j-th flow.
- If $g \in L_-$ has a simple pole, then $u \mapsto \tilde{u}$ is a Bäcklund transformation.

The VNLS soliton hierarchy (Fordy-Kulish 1983)

The splitting that gives VNLS hierarchy: Fix $\epsilon > 0$,

$$L = \{f : \{|\lambda| = \epsilon^{-1}\} \to GL(n+1, \mathbb{C}) \mid f(\bar{\lambda})^* f(\lambda) = \mathrm{I}\},$$
$$L_+ = \{f \in L \mid f \text{ extends holomorphically to } |\lambda| < \epsilon^{-1}\},$$
$$L_- = \{f \in L \mid f \text{ extends holomorphically to } \infty = |\lambda| > \epsilon^{-1}, f(\infty) = \mathrm{I}\}.$$

The vacuum sequence is: $\{a\lambda^j \mid j \geq 1\}$. Then the second flow constructed by this splitting and vacuum sequence is the VNLS. The third flow and the corresponding flow on $\mathbb{C}P^n$ are respectively

$$q_t = q_{xxx} + 4|q|^2 q_x,$$
$$\gamma_t = \nabla^2_{\gamma_x} \gamma_x - 4||\gamma_x||^2 \gamma_x.$$

BT and L_- action for SGE

- Let $s \in \mathbb{R} \setminus 0$, and $\pi \in gl(2, \mathbb{R})$ an orthogonal projection. Then
$$g_{is,\pi}(\lambda) = \mathrm{I} + \frac{2is}{\lambda - is}\pi \quad \in L_-.$$
- $u_f \mapsto u_{fg_{is,\pi}^{-1}}$ is the classical BT for the SGE.
- Given s_1, s_2, π_1, π_2, $\exists$ unique τ_1, τ_2 such that
$$g_{is_1,\tau_1} g_{is_2,\pi_2} = g_{is_2,\tau_2} g_{is_1,\pi_1},$$
and τ_i's can be computed algebraically from π_1, π_2. So Bianchi's Theorem follows directly form L_--action.

Wilson's μ function

Let $\rho : \hat{L} \to L$ be the central extension of L defined by the 2-cocycle

$$w(\xi, \eta) = \mathrm{Res}_\lambda \mathrm{tr}(\xi_\lambda \eta),$$

and $S_+ : L_+ \to \hat{L}$, $S_- : L_- \to \hat{L}$ group homorphism lifts of ρ.

Wilson's μ function $\mu : L \to \mathbb{C}^*$
is defined as follows: Given $f \in L$, factor

$$f = f_-^{-1} f_+ = g_+ g_-^{-1}, \quad f_\pm, g_\pm \in L_\pm,$$

then $S_-(f_-^{-1})S_+(f_+)$ and $S_+(g_+)S_-(g_-^{-1})$ lie in the same fiber $\rho^{-1}(f)$. Hence they differ by a non-zero complex number, which is defined to be $\mu(f)$. In other words,

$$S_-(f_-^{-1})S_+(f_+) = \mu(f) S_+(g_+) S_-(g_-^{-1}).$$

Tau function

The tau function given by $f \in L_-$ is

$$\tau_f(t_1, \ldots, t_N) = \mu(f^{-1} \exp(\sum_{i=1}^{N} t_i J_i)).$$

Let $J_j = a\lambda^j$. Factor

$$f^{-1}(\lambda) \exp(\sum_{j=1}^{n} a\lambda^j t_j) = E(t_1, \ldots, t_N, \lambda) M^{-1}(t_1, \ldots, t_N, \lambda)$$

with $E(t_1, \ldots, t_N, \cdot) \in L_+$ and $M(t_1, \ldots, t_N, \cdot) \in L_-$.

Theorem (Terng-Uhlenbeck 2016)
Given $f \in L_-$, then:

- $u_f = \mathrm{Res}_\lambda[a, M]$ is the solution of the VNLS hierarchy associated to f.
- We have
  (i) $y_j := (\ln \tau_f)_{t_1 t_j} = \mathrm{Res}_\lambda(\mathrm{tr}(MaM^{-1}a\lambda^j))$, for $1 \leq j \leq n$,
  (ii) u_f can be obtained from $y_1, \ldots, y_n$ and a system of odes.

Remark

- For the KdV hierarchy, it is known that $u_f = (\ln \tau_f)_{t_1,t_1}$.
- This is not the case for the NLS. If we write the solution $u_f = re^{i\theta}$, then

$$r^2 = (\ln \tau_f)_{t_1 t_1}, \quad \theta' = -\frac{(\ln \tau_f)_{t_1 t_2}}{(\ln \tau_f)_{t_1 t_2}}.$$

  So we can only recover u_f from τ_f up to a constant in S^1, but τ_f recovers the corresponding Schrödinger flow on S^2 uniquely.

Virasoro action on tau functions

The positive Virasoro algebra $\mathcal{V}_+$ is the algebra generated by $\{\xi_j | j \geq -1\}$ that satisfies

$$[\xi_j, \xi_k] = (k - j)\xi_{j+k}.$$

Our project on Virasoro actions was motivated by the remarkable work of Kontsevich: the tau function of the KdV hierarchy, that is fixed by the Virasoro action, gives the generating function of the quantum cohomology of a point. We constructed Virasoro actions on soliton hierarchies in an elementary way and hope their fixed points may be of some use.

Theorem (Terng-Uhlenbeck 2016)

- $\xi_j(f) = -(\lambda^{j+1} f_\lambda f^{-1})_- f$, $j \geq 1$, defines an action of $\mathcal{V}_+$ on L_-.
- The induced $\mathcal{V}_+$ action on $\chi = \ln \tau_f$ is given by

$$\zeta_\ell \chi = \sum_{j=1}^{N} j t_j \chi_{t_{j+\ell}} - \frac{1}{2} c_\ell(f), \quad \ell = -1, 0, 1,$$

$$\zeta_\ell \chi = \sum_{j=1}^{N} j t_j \chi_{t_{j+\ell}} + \sum_{j=1}^{\ell-1} \left(\chi_{t_j} \chi_{t_{\ell-j}} + \frac{1}{2} \chi_{t_j t_{\ell-j}} \right) - \frac{1}{2} c_\ell(f), \quad \ell \geq 2.$$

where $c_\ell(f) = \mathrm{Res}_\lambda(\lambda^{\ell+1} (f_\lambda f^{-1})^2)$ are constants.