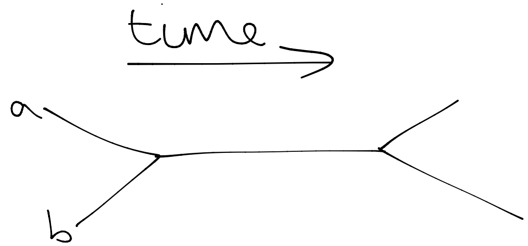
More physics

Ref [CK, §1]

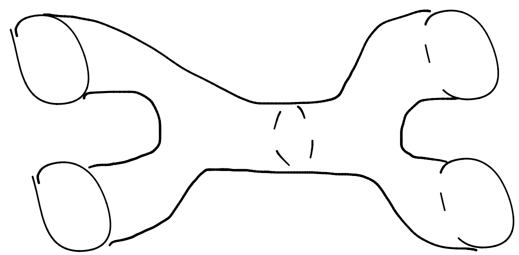
We return to discussing the

$$\boxed{\text{Super conformal field theory (SCFT)}}$$

An interaction between two particles a, b in spacetime M may be drawn like this



For two interacting strings, we may instead take a picture as follows



We therefore consider surfaces $\Sigma \subset M$.

The theory should not depend on choice of a coordinate system on Σ. This leads to a requirement that theory is invariant under a certain

$$\boxed{\text{Super conformal algebra } A \supset u(1) \times u(1)}$$

Taking generators for the latter subalgebra, and letting them act on the Hilbert space of the theory (i.e. the space of quantum states) we have that

$$\boxed{\begin{array}{l} (p,q) \text{ eigenspace} \longleftrightarrow H^q(\Lambda^p T) \\ (-p,q) \text{ eigenspace} \longleftrightarrow H^q(\Omega^p) \end{array}}$$

where we take T = tangent bundle on M
and Ω^p = p-forms on M

considering M as a complex manifold.

Recall we claimed an automorphism Φ of $\{SCFTs\}$.
This Φ exchanges the $(\pm p, q)$ eigenspaces in the representation above.

For a mirror pair X, X' we therefore expect

$$H^q(\Lambda^p T_X) \cong H^q(\Omega^p_{X'})$$

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But $\Lambda^p T_X \otimes \cancel{\omega_X} \cong \Omega^{3-p}_X$ so this gives

$\because X$ CY

$$H^{3-p,q}(X) \cong H^{p,q}(X')$$

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<u>Note</u> this agrees with the relations between $h^{p,q}$ claimed above.

Moduli of SCFTs

We finish our discussion of physics background, to explain further mirror symmetry predictions which will be the subject of later parts of the course. After this, we will introduce toric geometry, to give us a source of examples for constructions.

Recall Calabi-Yau 3-fold X has

$$\boxed{\text{complex structure } J, \text{ metric } g, \text{ Kähler form } \omega.}$$

Note ω determines class in $H^2(X,\mathbb{R})$

(this is deRham cohomology, but we drop subscript "dR")

$H^2(X,\mathbb{R})$ contains natural integral lattice, which may be identified with $H^2(X,\mathbb{Z}) \subset H^2(X,\mathbb{R})$ (cohom with $\mathbb{Z}$ coeffs)

A further structure which appears in the construction of the SCFT is:

$$\boxed{\text{"B-field" } B \in H^2(X,\mathbb{R})/H^2(X,\mathbb{Z})}$$

Combining B and ω we get

$$\boxed{\text{complexified Kähler class } B+i\omega \in H^2(X,\mathbb{C})/H^2(X,\mathbb{Z})}$$

Note: Why do we say these are in the quotient?

Because a surface $\Sigma \subset X$ corresponds to terms

$\exp(2\pi i (B+i\omega)\cdot[\Sigma])$ in the SCFT

where $[\Sigma] \in H_2(X,\mathbb{Z}) \subset H_2(X,\mathbb{C})$ is the class

of Σ in homology: these only depend on the class

of $B+i\omega$ in the quotient.

Now, given the data $X, J, g, B+i\omega$ there exists a SCFT.

Varying this data continuously, we would like to obtain a moduli space $\mathcal{M}_{SCFT}$, that is a space whose points correspond to different SCFTs. We expect that, locally,

$$\mathcal{M}_{SCFT} \cong \mathcal{M}_K \times \mathcal{M}_C$$

where points of $\mathcal{M}_K$ and $\mathcal{M}_C$ correspond to choices of $B+i\omega$ and J, respectively. We say

$\mathcal{M}_K$ = complexified/stringy Kahler moduli

$\mathcal{M}_C$ = complex structure moduli.

$\mathcal{M}_C$ is a classical object, much studied before mirror symmetry, and is locally isomorphic to $H^{2,1}(X)$

This is because deformations of complex structure are given by elements of $H^1(T_X)$ (see [H, §6.1])

and for Calabi-Yau X, $H^1(T_X) \cong H^1(\Omega^2_X) = H^{2,1}(X)$

$\mathcal{M}_K$, on the other hand, is special to mirror symmetry and does not yet have a full mathematical definition.

From the above description, we expect

$$\mathcal{M}_K \subset H^2(X,\mathbb{C})/H^2(X,\mathbb{Z})$$

to be locally isomorphic to $H^{1,1}(X)$, as the Kahler form ω gives a class in $H^{1,1}(X) \subset H^2(X,\mathbb{C})$.

Moduli <u>under mirror symmetry</u>

Some part of the SCFT depends only on $B+i\omega$, and some part only on J. In physics terminology, these are called <u>A-</u> and <u>B-model</u>.

Therefore we have

- A-model: determined by symplectic manifold (X, ω) and B-field
- B-model: determined by complex manifold (X, J)

Now take a mirror partner X'. By definition, X and X' have isomorphic SCFTs, so we have an identification

$$\mathcal{M}_{SCFT}(X) \cong \mathcal{M}_{SCFT}(X')$$

Locally, this isomorphism swaps $\mathcal{M}_K$ and $\mathcal{M}_C$ so that we have

$$\mathcal{M}_K(X) \cong \mathcal{M}_C(X') \text{ and } \mathcal{M}_C(X) \cong \mathcal{M}_K(X')$$

These are called the mirror maps, and can be calculated and studied.

Conceptually, they imply that mirror symmetry relates symplectic geometry on X to complex geometry on X'; and likewise with X and X' exchanged.

Note: The mirror map gives an indirect way to define $\mathcal{M}_K(X)$: we find a mirror X', and set $\mathcal{M}_K(X) = \mathcal{M}_c(X')$.

This approach is commonly taken.