Lecture 7: The Alexander Polynomial

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June 5, 2022

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Conway polynomial

In 1969, John Conway constructed a version of this polynomial by the skein relation:

$\nabla($  $) - \nabla($  $) = z\nabla($  $),$

$$\nabla(O) = 1$$

One can show that $\nabla(L)(t - t^{-1}) = \Delta(L)(t^2)$ and we obtain the following definition of the Alexander polynomial.

• The Alexander-Conway polynomial can be defined by skein relations:

$\Delta($  $) - \Delta($  $) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta($  $),$

$$\Delta(O) = 1.$$

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Alexander Matrix

Given an oriented link diagram L with n vertices, let us construct the Alexander matrix M(L) as follows.
Let us enumerate all crossings by natural numbers $1, \ldots, n$. In the general position, there exists precisely one arc outgoing from each crossing (if there are no separated cyclic arcs). It is easy to see that each knot isotopy class has such a diagram. So, we can enumerate outgoing arcs by integers from 1 to n, correspondingly. Now, we construct an incidence matrix, where a crossing corresponds to a row, and an arc corresponds to a column.
Suppose that no crossing is incident twice to one and the same arc (no loops). Then, each crossing (number i) is incident precisely to three arcs: passing through this crossing (number j), incoming (number k) and outgoing (number i).
In this case, the i−th row of the Alexander matrix consists of the three elements at places i, j, k. If the i−th crossing is positive, then $m_{i,i} = 1, m_{i,k} = -t, m_{i,j} = t - 1$. Otherwise we set $m_{i,i} = t, m_{i,k} = -1, m_{i,j} = 1 - t$.
Obviously, this matrix has determinant zero, because the sum of elements in each row equals zero.

Define the algebraic complement to $m_{i,j}$ by $\Delta_{i,j}$. Then the following theorem holds.

Theorem 2.1

All $\Delta_{i,j}$ coincide up to multiplication by $\pm t^k$.

Denote $\Delta_{i,j}$ by $\Delta(L)$.

Theorem 2.2

The function Δ defined on links (and normed properly) satisfies the following skein relation:

$\Delta($$) - \Delta($$) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta($$),$

$$\Delta(O) = 1.$$

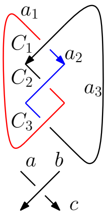
The Conway polynomial is obtained from the Alexander polynomial just by a variable change: $x = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$. Thus, the polynomial (denoted by C) satisfies the skein relation

C() − C() = x · C(),

$$C(O) = 1.$$

Conway first proved that this relation can be axiomatic for defining a knot invariant.

- As an example, we calculate the Alexander matrix of trefoil knot.



$ta + (1-t)b - c = 0$

$C_1 : ta_1 + (1-t)a_3 - a_2 = 0$
$C_2 : ta_3 + (1-t)a_2 - a_1 = 0$
$C_3 : ta_2 + (1-t)a_1 - a_3 = 0$

A minor $A_{(n-1)\times(n-1)}$

$$\begin{pmatrix} t & -1 & 1-t \\ -1 & 1-t & t \\ 1-t & t & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Alexander matrix A

$$det(A_{(n-1)\times(n-1)}) = \Delta(K)$$

Figure 1: The Alexander matrix of a trefoil knot and $\Delta(3_1) = -1 + \mathrm{t} - \mathrm{t}^2$

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The Wirtinger presentation

Now, let us find the system of relations for this group.
It is easy to see the geometrical connection between loops hooking adjacent edges (i.e., edges separated by an overcrossing edge). Actually, we have $b = cac^{-1}$, where c separates a and b; see Fig. 2.

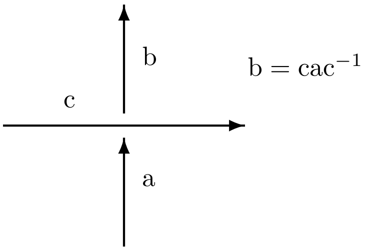

Figure 2: Relation for a crossing

The Wirtinger presentation

Let us show that all relations in the fundamental group of the complement arise from these relations.
Actually, let us consider the projection of a loop on the plane of $\bar{\mathrm{L}}$ and some isotopy of this loop. While transforming the loop, its written form in terms of generators changes only when the projection passes through crossings of the link, see Fig. 3.

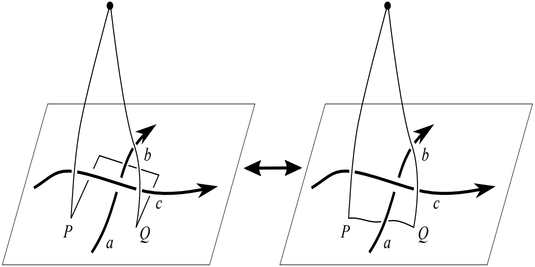

Figure 3: Isotopy generating relation

The Wirtinger presentation

Obviously, the loop shown on the left side of Fig. 3 generates the element $c^{-1}bc$, that on the right side is just a.
Thus, our presentation of the fundamental group of the link complement is

$$\pi_i(S^3 \backslash K) = \langle \{a_1, a_2, \cdots, a_n\} | r_n \rangle,$$

where a_i corresponds to an arc and $r_i : cac^{-1} = b$ coming from crossings as described in Fig. 2.

Definition 3.1

This presentation of the fundamental group for the knot complement is called the Wirtinger presentation.

The Wirtinger presentation

We present a way of calculating the fundamental group for arbitrary links. Consider a link L given by some planar diagram $\bar{L}$ and a point x "hanging" over this plane. Let us classify isotopy classes of loops outgoing from this point. It is easy to see (the proof is left to the reader) that one can choose generators in the following way. All generators are classes of loops outgoing from x and hooking the arcs of $\bar{L}$ turning according to the right–hand screw rule; see Fig. 4.

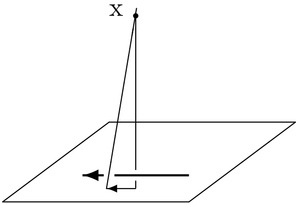

Figure 4: Loops corresponding to edges

The standard description of the Alexander polynomial $\Delta(K)$ is closely related to the standard Wirtinger presentation of the fundamental group. With a link diagram, we associate generators to arcs, and take the relation $bab^{-1} = c$ in the case of groups and $c = ta + (1 - t)b$ in the Alexander module. The same group and module can be presented from another point of view. Here we follow closely [5].

Namely, let D be a link diagram on the plane P with n crossings. Then by Euler characteristic argument this diagram tiles the sphere (1-point compactification of the plane) into $n + 2$ cells, including the infinite one. Thus, we have $n + 1$ finite regions $r_1, \cdots, r_{n+1}$.

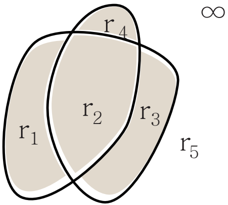

Figure 5: Since the trefoil knot has 3 crossings, the plane separated by 5 regions with 4 finite regions

The Dehn presentation

We present another way of calculating the fundamental group for arbitrary links, which is called the Dehn presentation. In the Dehn presentation, regions, in which the diagram divides the plane, correspond to the generators and the generating relations come from crossings as described in Fig. 6.

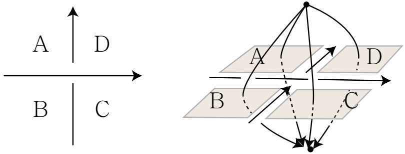

Figure 6: $r(c) = AB^{-1}CD^{-1}$

Then the knot group can be presented by

$$\pi_1(\mathbb{R}^3 \backslash K) = \langle \{\text{Regions}\} | r(c) = 1 \rangle.$$

Let us fix two points A and B over the projection plane and under the projection plane. We denote the link itself by L and we may think that it lies in the neighbourhood of P.
We may fix a path p from B to A pickling the plane P in the infinite region.
Then the group $\pi_1(\mathbb{R}^3 \setminus \mathrm{L})$ can be presented in the following way. We take A to be the base point. Now, with all finite regions $\mathrm{r}_1, \cdots, \mathrm{r}_{\mathrm{n}+1}$ we associate generators $\mathrm{g}_1, \cdots, \mathrm{g}_{\mathrm{n}+1}$. Each generator g_j pickles the plane in the finite region from A to B and then goes back through the infinite region as indicated in Fig. 7.

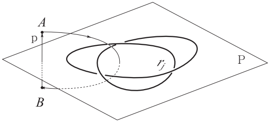

Figure 7: Dehn generator g_j

Then, one can easily see that for each crossing c where some four regions r_i, r_j, r_k, r_l meet, the relation $g_i g_j^{-1} g_k g_l^{-1} = 1$ holds; see Fig. 8.

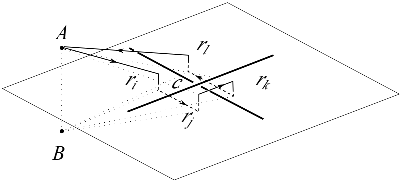

Figure 8: Crossing relation $g_i g_j^{-1} g_k g_l^{-1} = 1$

One can easily check that this gives rise to a presentation of the fundamental group:

$$\pi_1(\mathbb{R}^3 \setminus L) = \langle g_1, \dots, g_{n+1} \,|\, g_i g_j^{-1} g_k g_l^{-1} = 1 \text{ for all crossings } c \rangle. \quad (1)$$

This presentation is called the Dehn Presentation [Dehn2].

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Now, let us pass to the definition of the Alexander module and the Alexander polynomial via Dehn's presentation.
Four regions meet locally at a given crossing. Letting these be labeled generically $\{A, B, C, D\}$, as shown in Figure 9, we have an equation in the Alexander module

$$xA - xB + C - D = 0 \tag{2}$$

to that crossing.

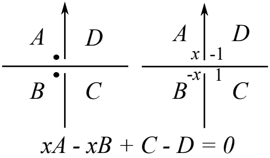

Figure 9: Alexander labeling

Here A, B, C, D go cyclically around the crossing, starting at the top dot. In this way the two regions containing the dots give rise to the two occurrences of x in the equation.

If some of the regions are the same at the crossing, then the equation is simplified by that equality.
For example, if $A = D$ then the equation becomes $xA - xB + C - A = 0$. Each crossing in a diagram K gives an equation involving the regions of the diagram.
Let us associate the Dehn presentation Alexander matrix to the diagram K — a matrix M_K whose rows correspond to the crossings of the diagram, and whose columns correspond to the regions of the diagram.

Each nodal equation gives rise to one row of the matrix where the entry for a given column is the coefficient of that column (understood as designating a region in the diagram) in the given equation. If R and R' are adjacent regions, let $M_K[R, R']$ denote the matrix obtained by deleting the corresponding columns of M_K.

Finally, define the Alexander polynomial $\Delta_K(x)$ by the formula

$$\Delta_K(x) \doteq \mathrm{Det}(M_K[R, R']). \quad (3)$$

The notation $a \doteq b$ means that $b = \pm x^n a$ for some integer n. It is proved in [1], that this polynomial is well-defined, independent of the choice of adjacent regions and invariant under the Reidemeister moves up to $\doteq$.

In Figure 10 we have shown the calculation of the Alexander polynomial of the trefoil knot using this method.[1]

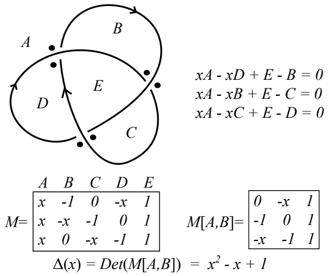

Figure 10: The Alexander polynomial

In Fig. 10 we show the diagram of the knot, the labelings and the resulting full matrix and the square matrix resulting from deleting two columns corresponding to a choice of adjacent regions. Computing the determinant, we find that the the Alexander polynomial of the trefoil knot is given by the equation $\Delta \doteq x^2 - x + 1$.

[1] Figures 10—14 are borrowed from Kauffman [Kau3].

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Alexander matrix and single-circle states

In the Alexander determinant expansion the choice of a row by a column corresponds to a region choosing a crossing in the link diagram. The only crossings that a region can choose giving a non-zero term in the determinant are the crossings in the boundary of the given region.

Thus the terms in the expansion of $\mathrm{Det}(M[A, B])$ are in one-to-one correspondence with decorations of the flattened link diagram (i.e. we ignore the over and under crossing structure) where each region (other than the two deleted regions corresponding to the two deleted columns in the matrix) labels one of its crossings.

We call these labeled flat diagrams the states of the original link diagram.

See Figure 11 for a list of the states of the trefoil knot. In this figure we show the states and the corresponding matrix forms with columns choosing rows that correspond to each state.

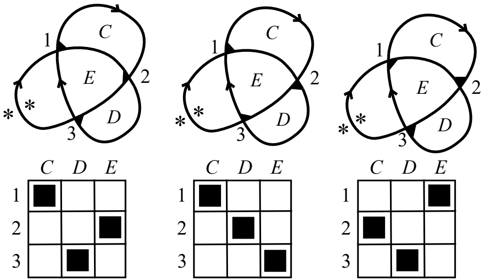

Figure 11: States with markers

At this point we have an almost complete combinatorial description of Alexander's determinant. The only thing missing is the permutation signs. One can pick up the permutations from the state labeling, but there is a better way. Call a state marker (label at a crossing as shown in Figure 11) negative if it labels a quadrant where both oriented segments point toward the crossing; see Fig. 12.

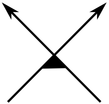

Figure 12: A negative marker

Let S be a state of the diagram K. Set $(-1)^{b(S)}$ where b(S) is the number of negative markers in the state S. Then it turns out that with up to one global sign ϵ depending on the ordering of crossings and regions, we have

$$(-1)^{b(S)} = \epsilon \operatorname{sgn}(\sigma(S)),$$

where $\sigma(S)$ is the permutation of crossings induced by the choice of ordering of the regions of the state. This gives a purely diagrammatic access to the sign of a state and allows us to write

$$\Delta_K(x) \doteq \Sigma_S \langle K|S\rangle (-1)^{b(S)}, \tag{4}$$

where S runs over all states of the diagram for a given choice of deleted adjacent regions, and $\langle K|S\rangle$ denotes the product of the Alexander nodal labels at the quadrants indicated by the state labels in the state S. We call $\langle K|S\rangle$ the product of the vertex weights. Thus we have a precise reformulation of the Alexander polynomial as a state summation.

Let us mention that every term in the Dehn presentation Alexander matrix corresponds to a single-circle state, see Fig. 13.

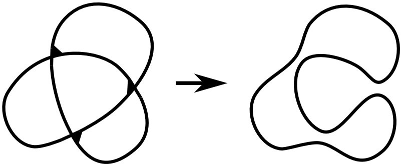

Figure 13: A single circle state

In Figure 14 we illustrate the calculation of the Alexander polynomial of the trefoil knot using this state summation. Here we show the contributions of each state to a product of terms and in the polynomial we have followed the state summation by taking into account the number of negative markers in each state. Thus we get

$$(-1)^{\mathrm{b(S)}} = \epsilon \ \mathrm{sgn}(\sigma(\mathrm{S})).$$

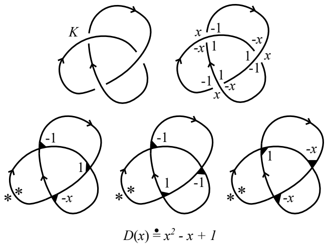

Figure 14: State sum calculation of Alexander polynomial

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Seifert surfaces

Definition 6.1

Let L be an oriented link. A Seifert surface for the link L is a closed compact oriented two–dimensional surface in $\mathbb{R}^3$, whose boundary is the link L and the orientation of the link L is induced by the orientation of the surface. For a given link diagram, a Seifert surface consists of finitely many discs, called Seifert discs, which are connected by bands corresponding to crossings.

Denote the genus of a Seifert surface F(L) for the link L by g(F(L)). The graph $\Gamma(L)$ consisting of vertices corresponding to Seifert circles and edges corresponding to bands is called Seifert graph.
Note that $2g(F(L)) - \mu(L) + 1 = 1 - d + c = f - 1$, where d is the number of Seifert discs, c is the number of crossings and f is the number of faces of $\Gamma(L)$.

Theorem 6.2

For each link in $\mathbb{R}^3$, there exists a Seifert surface of it.

Seifert surfaces

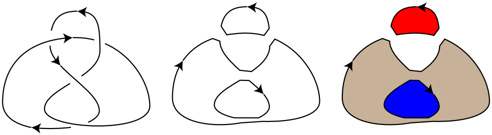

Figure 15: Seifert circles and discs

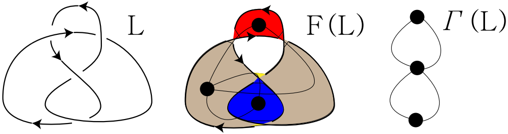

Figure 16: Seifert surface F(L) and Seifert graph Γ(L). We obtain $2g(F(L)) - 1 + 1 = 1 - 3 + 2 = 3 - 1 = 2$

Seifert matrix

Let F(L) be a Seifert surface and Γ(L) a Seifert graph. We take simple closed curve $\{\alpha_i\}_{i=1}^{f-1}$ on F(L) such that α_i corresponds to the boundary of faces of Γ_L except for the face containing ∞, see Fig. 17. Consider $F(L) \times [0,1]$ and take $\alpha_i = \alpha_i \times \{0\}$ and $\alpha_i^\sharp = \alpha_i \times \{1\}$. A $(f-1) \times (f-1)$ matrix M_L defined by

$$M_L = [\mathrm{lk}(\alpha_i, \alpha_j^\sharp)]_{i,j}$$

is called a Seifert matrix.

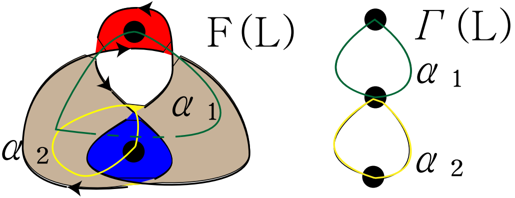

Figure 17:

Theorem 6.3

Two Seifert matrices, obtained from two equivalent knots (or links) can be changed from one to the other by applying, a finite number of times, the following two operations, Λ_1 and Λ_2, and their inverse:

$$\Lambda_1 : \mathrm{M}_1 \to \mathrm{P}\mathrm{M}_1\mathrm{P}^{\mathrm{T}},$$

where P is an invertible integer matrix, with $\det(\mathrm{P}) = \pm 1$, and P^{T} denotes the transpose matrix of P.

$$\Lambda_2 : \mathrm{M}_1 \to \mathrm{M}_2 = \begin{pmatrix} \mathrm{M}_1 & * & \mathrm{O} \\ * & 0 & 1 \\ \mathrm{O} & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} \mathrm{M}_1 & * & \mathrm{O} \\ * & 0 & 0 \\ \mathrm{O} & 1 & 0 \end{pmatrix}$$

where $*$ is block matrices with arbitrary integer entries and O is a zero matrix.

Alexander polynomial

Let L be a link and M_L a Seifert matrix. The determinant $\det(M_L + tM_L)$ of $M_L + tM_L$ is called the Alexander polynomial, where t is an indeterminate. We denote it by $\Delta(L)$. As a Corollary of Theorem 6.3, we obtain the following theorem:

Theorem 6.4

The Alexander polynomial $\Delta(L)$ is an invariant under Reidemeister moves.

Theorem 6.5

If for a knot K the Alexander polynomial is trivial then K is topologically slice, i.e., there exists a disc $D^2 \in B^4$ which is locally flat such that $\partial D^2 = K \in S^3 = \partial B^4$.

Definition 6.6

The span $|\Delta(K)|$ of $\Delta(K)$ is the difference between its highest and lowest degrees of $\Delta(K)$.

Theorem 6.7

The Seifert genus of a knot K, g(K), is bounded below by the span of the Alexander polynomial:

$$g(K) \geq \frac{1}{2}|\Delta(K)|.$$

Proof. Let F be a connected, minimal-genus, spanning surface for K and let $r = 2g$. A basis for $H_1(X)$ has r generators, so the Seifert matrix for F is an $r \times r$ square matrix such that each entry has t or t^{-1}. Thus, the difference between the largest degree for t and the smallest degree is less than or equal to r. Hence, we obtain that

$$|\Delta(K)| \leq r = 2g,$$

and proof is completed. □

Definition 6.8

The span $|\Delta(K)|$ of $\Delta(K)$ is the difference between its highest and lowest degrees of $\Delta(K)$.

Theorem 6.9

The Seifert genus of a knot K, $g(K)$, is bounded below by the span of the Alexander polynomial:

$$g(K) \geq \frac{1}{2}|\Delta(K)|.$$

Proof. Let F be a connected, minimal-genus, spanning surface for K and let $r = 2g$. A basis for $H_1(X)$ has r generators, so the Seifert matrix for F is an $r \times r$ square matrix such that each entry has t or t^{-1}. Thus, the difference between the largest degree for t and the smallest degree is less than or equal to r. Hence, we obtain that

$$|\Delta(K)| \leq r = 2g,$$

and proof is completed. □

Fox-Milnor condition

Theorem 6.10 (Fox-Milnor condition)

The Alexander polynomial of a slice knot factors as a product $f(t)f(t^{-1})$ where $f(t)$ is some integral Laurent polynomial.

One can find this theorem in [2].

Theorem 6.11

For every knot, the span of its Alexander polynomial is greater than or equal to twice its locally flatly slice genus.

For topological slice knots, we have a beautiful theorem, proved by M. Freedman:

Theorem 6.12 ([4])

For a knot K if the value of the Alexander polynomial is 1, then the knot K is locally flatly slice.

Contents

Exercises

1. Show that all definitions of the Alexander polynomial coincides up to multiplication by $\pm t^n$.
2. Prove that $M[A, B]$ does not depend on the choice of B.
3. Calculate the genera of knots 3_1 and 4_1.
4. Prove that for n-component link all coefficients c_{2k+1} of the Conway polynomial are trivial if n is odd, and all coefficients c_{2k} are trivial if n is even.

A good report

Dror Bar-Natan gave a report named “I Still don’t Understand the Alexander Polynomial” at “Mathematical colloquium in BMSTU” (https://www.ktrt-seminars.com) (you can find the report and ppt at http://drorbn.net/mo21)

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