2021-9-13 Kähler geometry

$\underset{\sim}{y} = A \underset{\sim}{x}$, $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$\longrightarrow$

清華園

二校门

column vector

western culture

園華清 eastern culture

$\longleftarrow$

$\underset{\sim}{x} A = \underset{\sim}{y}$, $(x_1 \cdots x_n) A = (y_1 \cdots y_n)$

row vector

We employ the western culture, and we express vectors as column vectors.

(Both are possible.)

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, $\dim = n$.

Let $e_1, \dots, e_n$ be a basis.

Then

$$\mathbb{X} = x^1 e_1 + \cdots + x^n e_n$$

But we want to express a vector as a column vector

$$\mathbb{X} = (e_1 \cdots e_n) \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$$

$$= e x$$

Note that we used upper index for coefficients, and lower index for basis.

This is a standard notation in differential geometry and physics.

Einstein summation convention

$$\mathbb{X} = x^i e_i = e_i x^i$$

omitting $\sum_{i=1}^{n}$ Understand that the sum is taken if upper and lower index appear at the same time.

Take the dual basis $e^1, \dots, e^n$ of V^*

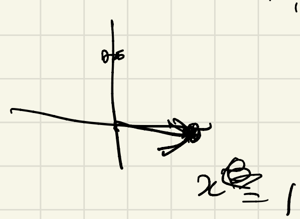
w.r.t. $e_1, \dots, e_n$.

$$\langle e^i, e_j \rangle = \delta^i{}_j \quad \text{Kronecker delta}$$

Take $\alpha \in V^*$ is expressed as

$$\alpha = \alpha_i e^i \left(= \sum_{i=1}^{n} \alpha_i e^i \right).$$

$$x = x^i e_i$$



$f_1 \cdots f_n$

Let $f_1, \dots, f_n$ be another basis

Then $\exists P$ non-singular matrix s.t.

$$(f_1 \cdots f_n) = (e_1 \cdots e_n) P, \quad P = (p^i{}_j)$$

$$f_j = e_i p^i{}_j = p^i{}_j e_i \quad \left(= \sum p^i{}_j e_i\right)$$

$$P \in \mathrm{End}(V) = V \otimes V^*$$

$$\parallel \quad p^i{}_j \, e_i \otimes e^j$$

$P = \cancel{(p^i_j)}$

$= (p^i{}_j)$

tex notation

p^i{}_j

Let $T : V \to V$ be a linear map

$$T(e_j) = \sum_{i=1}^{n} a^i{}_j \, e_i$$

$$= e_i \, a^i{}_j$$

$$T(e_1 \cdots e_n) = (T(e_1), \cdots, T(e_n))$$

$$= (e_1 \cdots e_n) A$$

A is called the matrix expression of the linear map T w.r.t. $e_1, \cdots, e_n$.

$$T(f_1 \cdots f_n) = (T(f_1) \cdots T(f_n))$$

$$= (f_1 \cdots f_n) A'$$

What is the relation of A and A' ?

Answer $\quad A' = P^{-1}AP.$

∵

$$T(f_1 \cdots f_n) = (f_1 \cdots f_n)A'$$

$$\| \qquad = (e_1 \cdots e_n)P \cdot A'$$

$$T((e_1 \cdots e_n)P) = (T(e_1) \cdots T(e_n))P$$

$$= (e_1 \cdots e_n)AP$$

$$PA' = AP \qquad \therefore \quad P = A^{-1}PA'.$$

∵

Following backwards this arguments, we see that if $A' = P^{-1}AP$ then A and A' are expressions of a same linear map T with respect to different bases.

Vector bundles.

Let M be a smooth manifold of dim n.

Definition

$\pi : E \to M$ between smooth manifold E and M is said to be a real or complex vector bundle of rank r if

(i) $\dim E = n+r$. (or $n+2r$)

(ii) π is a smooth map and the rank of $d\pi$ is n everywhere (maximal rank)

(iii) There is an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of M with diffeo

$$\varphi_\lambda : \pi^{-1}(U_\lambda) \longrightarrow U_\lambda \times \mathbb{R}^r \quad (\text{or } U_\lambda \times \mathbb{C}^r)$$

such that

(iii a) for the projection $P_\lambda : U_\lambda \times \mathbb{R}^n \longrightarrow U_\lambda$

$\underline{\pi = P_\lambda \circ \varphi_\lambda}$ i.e. $\qquad$ <u>local triviality</u>

$$
\begin{tikzcd}
E \supset \pi^{-1}(U_\lambda) \arrow[rr] \arrow[dr, "\pi"'] & \arrow[d, phantom, "\circlearrowright"] & U_\lambda \times \mathbb{R}^r \arrow[dl, "P_\lambda"] \\
& p \in U_\lambda \subset M &
\end{tikzcd}
$$

$$\varphi_\lambda\left(\pi^{-1}(p)\right) = P_\lambda^{-1}(p)$$

(iii b) When $U_\lambda \cap U_\mu \neq \emptyset$

$$\varphi_\lambda \circ \varphi_\mu^{-1} : (U_\lambda \cap U_\mu) \times \mathbb{R}^r \longrightarrow (U_\lambda \cap U_\mu) \times \mathbb{R}^r$$

is expressed as

$$\underline{\varphi_\lambda \circ \varphi_\mu^{-1}}(p, x) = (p, \underline{\varphi_{\lambda\mu}(p)}\, x)$$

$$\varphi_{\lambda\mu} : U_\lambda \cap U_\mu \longrightarrow GL(r, \mathbb{R})$$

$\mathbb{C}$

Remark

1. $\{\varphi_{\lambda\mu}\}_{\lambda,\mu\in\Lambda}$ are called the transition functions.

2. $\pi^{-1}(p)$ is called a fiber, and has a structure of vector space because

$$\left(\varphi_\lambda|_{\pi^{-1}(p)}\right)^{-1} : \underline{p\times\mathbb{R}^r} \longrightarrow \pi^{-1}(p)$$

and the vector space structure $p\times\mathbb{R}^r$ is independent of $\lambda\in\Lambda$. by (iiib)

3. Replacing $\mathbb{R}$ by $\mathbb{C}$ and <u>assuming</u>
   - (a) M is a complex manifold.
   - (b) E is also a complex manifold.
   - (c) all maps are holomorphic

   we say E is a holomorphic vector bundle.

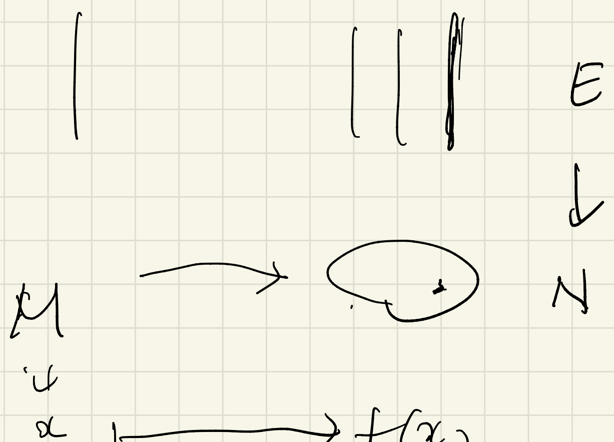
4. Typical examples of vector bundles are tangent bundles and cotangent bundles.

and their tensor products.

5. When $r=1$, E is called a line bundle. often denoted by L.

6. $\pi : E \longrightarrow N$ vector bundle over N, and $f : M \longrightarrow N$ smooth, then we have the pull-back bundle

$$f^*E = \{ (x, v) \in M \times E \mid v \in \pi^{-1}(f(x)) \}$$



7. E is called a trivial bundle if it is a pull-back bundle of the product $\underline{M \times \mathbb{R}^r}$ or $M \times \mathbb{C}^r$

8. $P := \{(e_1, \cdots, e_n) \mid \text{basis of } T_pM,\ p \in M\}$

P is called the frame bundle.

(A basis $(e_1, \cdots, e_n)$ is called a "frame".)

The frame bundles is a typical example of the "principal bundles".

In this case, the structure group is $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$.

As we saw above, the structure group $GL(n, \mathbb{C})$ acts on P from the right.

$$(e_1' \cdots e_n') = (e_1 \cdots e_n)\, P$$

If we choose $xA = y$, the structure group acts from the left.

Majority of people choose the convention of the right action of the structure group.