Donaldson's theorem and Furuta's theorem

Theorem (Donaldson) Let X be a smooth 4-mfd with $b^+(X)=0$
Then $Q_X \cong -I$
(If $\pi_1(X)=1$, then by Freedman, $X \cong_{top} \# \overline{\mathbb{CP}}^2$.)

Donaldson's original proof: Study the ASD Yang-Mills equation
$F_A^+=0$, where A is a connection on $SU(2) \hookrightarrow P \to X$ with
$c_2(P)[X]=1$.

We will give a Seiberg-Witten proof:
First, if $b^1(X) \neq 0$, pick embedded loops $\gamma_1, \gamma_2, \cdots \gamma_{b_1}$
that is a basis $H_1(X;\mathbb{R})$.
Do surgery $X' = (X \setminus \underbrace{\bigsqcup_i \nu(\gamma_i)}_{\sqcup S^1 \times S^2}) \cup (\bigsqcup_i D^2 \times S^2)$
Then $Q_{X'} = Q_X$ and $b_1(X')=0$.
So we assume $b_1(X)=0$ WLOG.

Take $\mathfrak{s} \in \mathrm{Spin}^{\mathbb{C}}(X)$, consider the Seiberg-Witten map $\widetilde{SW}$

$$\{\mathrm{spin}^{\mathbb{C}}\text{-connections}\} \oplus \Gamma(S^+) \longrightarrow \Omega^2_+(X;i\mathbb{R}) \oplus \Omega^0_0(X;i\mathbb{R}) \oplus \Gamma(S^-)$$
$$(A, \phi) \longmapsto (F_{A^t}^+ - \rho^{-1}(\phi\phi^*)_0,\ d^*(A-A_0),\ D_A\phi)$$

Here $\Omega^0_0(X) = \{\beta \in \Omega^0(X)$ s.t. $\int \beta\, d\mathrm{vol} = 0$.

Assumption expected dim of $\mathcal{M}_{SW}$

- Suppose $d(\mathfrak{s}) = 2\,\mathrm{ind}_{\mathbb{C}}(\not{D}) + \underbrace{\overset{\overset{0}{\|}}{\mathrm{ind}_{\mathbb{R}}(\Omega^1 \to \Omega^0_0 \oplus \Omega^2_+)}}_{b^1 - b^+} - 1 \geq 0$

Then $\mathrm{ind}_{\mathbb{C}}(\not{D}) = \dfrac{c_1(\mathfrak{s})^2 + b_2(X)}{8} \geq 1$.

We can use this to prove a transversality result:

For generic $\eta \in \Omega^2_+(X; i\mathbb{R})$, $\widetilde{SW}^{-1}(\eta, 0, 0)$ is a smooth manifold of dimension $d(\mathfrak{s}) + 1 \rightsquigarrow$ even

$\mathcal{M}_{SW} = \widetilde{SW}^{-1}(\eta) / G_h$

$G_h = S^1$

Claim: $\widetilde{SW}^{-1}(\eta, 0, 0)$ contains a unique reducible point.

proof: $(A_0 + \alpha, 0) \in \widetilde{SW}^{-1}(\eta, 0, 0)$

$\iff d^+\alpha = \eta$ and $d^*\alpha = 0$

Since $b^+ = b_1 = 0$, the operator $d^+ \oplus d^* : \Omega^1 \to \Omega^2_+ \oplus \Omega^0_0$ is an isomorphism. So $\forall \eta$, $\exists!$ solution α. Claim proved.

Recall that $\widetilde{SW}^{-1}(\eta, 0, 0)$ is acted by

$G_h = \{\text{harmonic } X \to U(1)\} \overset{H^1(X;\mathbb{Z}) = 0}{=} \{\text{constant } X \to U(1)\} = S^1$

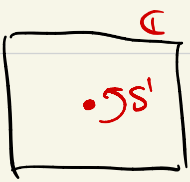
So $\widetilde{\mathcal{M}} := \widetilde{SW}^{-1}(\eta, 0, 0)$ satisfies:

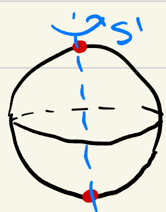
(1) $\dim \widetilde{\mathcal{M}} = 2k > 0$ (2) $\widetilde{\mathcal{M}}$ is compact, closed.

(3) $\widetilde{\mathcal{M}}$ has an S^1-action with a unique fixed point a

(4) The S^1-action on $\widetilde{\mathcal{M}} - \{a\}$

Claim: Such $\widetilde{\mathcal{M}}$ doesn't exist.





Proof: $\tilde{M}^\circ = \tilde{M} - \nu(a)$ $\quad \partial \tilde{M}^\circ = S^{2k-1}$

$S^1 \hookrightarrow \tilde{M}^\circ \to \tilde{M}^\circ/S^1 \Rightarrow \mathbb{C} \hookrightarrow L \to \tilde{M}^\circ/S^1$

$L|_{\partial(\tilde{M}^\circ/S^1)}$ is associated to $S^1 \hookrightarrow S^{2k-1} \to \mathbb{CP}^{k-1}$

so $c_1(L)|_{\partial(\tilde{M}^\circ/S^1)} =$ generator of $H^2(\mathbb{CP}^{k-1})$.

so $c_1(L)^{k-1} \cdot [\partial(\tilde{M}^\circ/S^1)] \neq 0$

But $[\partial(\tilde{M}^\circ/S^1)] = 0 \in H_{2k-2}(\tilde{M}^\circ/S^1)$ contradiction.

What's going wrong? $d(\mathfrak{s})$ must be < 0

so for all $\mathfrak{s} \in \mathrm{spin}^c(X)$, $c(\mathfrak{s})^2 + b_2(X) \leq 0$

so for all $\nu \in \mathrm{char}(X) \subset H^2(X;\mathbb{Z})/\mathrm{tors}$, we have

$Q_X(\nu,\nu) + b_2(X) \leq 0$

Theorem (Elkies) Let $Q: \mathbb{Z}^m \otimes \mathbb{Z}^m \to \mathbb{Z}$ be a negative definit, unimodular form.

Let $\mathrm{char}(Q) = \{\nu \in \mathbb{Z}^m \mid Q(\nu,\omega) \equiv Q(\omega,\omega) \bmod 2\}$.

Suppose $Q(\nu,\nu) + m \leq 0 \;\; \forall \nu \in \mathrm{char}(Q)$.

Then $Q \cong -I$

Proof of Furuta's theorem

($\frac{10}{8}$-thm) ($\pi_1(X)=1$ spin $\Leftrightarrow Q_X$ is even)

Theorem. Let X be a spin 4-mfd s.t. $b_2(X)\neq 0$.

$$b_2(X)\geq \frac{10}{8}|\sigma(X)|+2. \quad \left(\tfrac{11}{8}\text{-conj } b_2(X)\geq \tfrac{11}{8}|\sigma(X)|\right)$$

Recall: X is spin $\Rightarrow Q_X$ is even $\Rightarrow Q_X\cong P\cdot E_8\oplus q\begin{pmatrix}0&1\\1&0\end{pmatrix}$

so $\frac{11}{8}$-conjecture: $q\geq \frac{3}{2}|P|$

Furuta's theorem: $q\geq |P|+1$ (unless $P=q=0$)

Now we start the proof:

$\mathbb{H}=\{a+bi+cj+dk \mid a,b,c,d\in\mathbb{R}\}$ $i^2=j^2=k^2=-1$ $ij=-ji=k$

$Sp(1)=\{h\in\mathbb{H} \mid |h|=1\}\cong SU(2)=\{A\in M_{2\times 2}(\mathbb{C}) \mid AA^*=1\ \det(A)=1\}$

$1\longleftrightarrow \begin{bmatrix}1&0\\0&1\end{bmatrix}$, $i\longleftrightarrow\begin{bmatrix}0&i\\i&0\end{bmatrix}$

$j\longleftrightarrow\begin{bmatrix}0&-1\\1&0\end{bmatrix}$, $k\longleftrightarrow\begin{bmatrix}i&0\\0&-i\end{bmatrix}$

$SO(4)=\underset{\text{unit sphere in }\mathbb{H}}{\mathrm{Iso}^+(S(\mathbb{H}))}=\underset{\text{left multiplication}}{Sp(1)}\times \overset{\text{right multiplication}}{Sp(1)}/(-1,-1)$

so $Spin(4)=Sp(1)\times Sp(1)$

It has a complex representation $Spin(4)\xrightarrow{\rho}\mathrm{End}_{\mathbb{C}}(\mathbb{H}\oplus\mathbb{H})$

where $\underset{Spin(4)}{(\alpha,\beta)}(h_1,h_2):=(\alpha h_1,\beta h_2)$ $\mathrm{End}_{\mathbb{H}}(\mathbb{H}\oplus\mathbb{H})$

(Here $\mathbb{C}$ acts as right multiplication)

This is actually a quaternionic representation.

(i.e. the action commutes with the right multiplication with $\mathbb{H}$)

$e_1 \quad e_2 \quad e_3 \quad e_4 \quad \rightsquigarrow \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix} \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}$

So given a spin-structure $Spin(4) \hookrightarrow P \to X$

we get an associated quaternion bundle $S = S^+ \oplus S^-$

We can actually extend the action ρ to

$$\rho : Cl(T_* X) \longrightarrow End(S^+ \oplus S^-)$$

Also, note $P \xrightarrow{2:1} Fr$ so ∇^{LC} on Fr can be lifted to a unique connection on P (hence on S). This is called the spin connection A_0.

We can define the $\boxed{\text{spin Dirac operator}}$

$$\not{D}_{A_0} : \Gamma(S^+) \longrightarrow \Gamma(S^-) \qquad \phi \longmapsto \sum_{i=0}^{3} \rho(e_i) \cdot (\nabla_{A_0})_{e_i} \phi$$

Atiyah-Singer index theorem

$$ind_{\mathbb{C}}(\not{D}_{A_0}) = \frac{c_1(S^+)^2 - \sigma(X)}{8} = -\frac{\sigma(X)}{8}$$

Note: $\not{D}_{A_0}$ is quaternionic linear.

so $\underline{ind_{\mathbb{C}}(\not{D}_{A_0}) = 2 \cdot ind_{\mathbb{H}}(\not{D}_{A_0})}$ is even.

So we have proved

Theorem (Rokhlin) For any spin 4-mfd, $16 \mid \sigma(X)$.

Furuta's proof is a nonlinear version of this

First, spin structure $\Rightarrow$ spin$^{\mathbb{C}}$ structure

$\mathrm{spin}^{\mathbb{C}}(4) = \mathrm{spin}(4) \times U(1)/\mathbb{Z}_2$ so

$\mathrm{spin}(4) \hookrightarrow P \to X \quad \Rightarrow \quad \mathrm{spin}^{\mathbb{C}}(4) \hookrightarrow (P \times U(1))/\mathbb{Z}_2 \to X$

In this case, the spinor bundle $S^{\pm}$ is exactly the quatenionic bundle we have mentioned.

So we have the (extended) Seiberg-Witten map

$$\widetilde{SW} : \Omega^1(X; i\mathbb{R}) \oplus \Gamma(S^+) \longrightarrow \Omega^0_0(X; i\mathbb{R}) \oplus \Omega^2_+(X; i\mathbb{R}) \oplus \Gamma(S^-)$$

$$(\alpha, \phi) \longmapsto (d^*\alpha, d^+\alpha, \not{D}^+_{A_0}\phi) + (0, -\rho^{-1}(\phi\phi^*)_0, \rho(\alpha)\phi)$$

Three key properties of $\widetilde{SW}$: (Assume $H^1(X)=0$ by surgery.)

(1) $\widetilde{SW}^{-1}(0)$ is compact

(2) $\widetilde{SW} = L + Q$, where L is 1-st order linear elliptic operator

Q is 0-th order, nonlinear operator.

(3) $\widetilde{SW}$ is Pin(2)-equivariant.

Here $\mathrm{Pin}(2) = \{e^{i\theta}\} \sqcup \{j \cdot e^{i\theta}\} \subset Sp(1) \subset \mathbb{H}$

$= S^1 \sqcup jS^1$ with $je^{i\theta}j^{-1} = e^{-i\theta}$ $j^2 = -1$

The action is given by

$e^{i\theta}(\alpha) := \alpha \qquad j(\alpha) := -\alpha \qquad \alpha \in \Omega^1 \cup \Omega^2_+ \cup \Omega^0_0$

$e^{i\theta}(\phi) := \phi \cdot e^{i\theta} \qquad j(\phi) := \phi \cdot j$

The S^1-symmetry is the old gauge symmetry $\mathcal{G}_n = S^1$

j-symmetry : $\widetilde{SW}_{\mathfrak{s}} \xrightarrow{\text{conjugation}} \widetilde{SW}_{\bar{\mathfrak{s}}} \xrightarrow{\bar{\mathfrak{s}} = \mathfrak{s}} \widetilde{SW}_{\mathfrak{s}}$. ← since $\mathfrak{s}$ is spin.

Recall $\|f\|^2_{L^2_k} := \|f\|^2_{L^2} + \|f'\|^2_{L^2} + \cdots + \|f^{(k)}\|^2_{L^2}$ Sobolev norm

can be generalized to $\Omega^*(X)$ and $\Gamma(S^{\pm})$

Let $H_0 = L^2_{k+1}(\Omega^1(X; i\mathbb{R}) \oplus \Gamma(S^+))$

$H_1 = L^2_k(\Omega^0_0(X; i\mathbb{R}) \oplus \Omega^2_+(X; i\mathbb{R}) \oplus \Gamma(S^-))$

Hilbert spaces
Sobolev completion
$k \geq 3$

Then $\widetilde{SW} = L + Q : H_0 \to H_1$ --- ⊛

L ↑ Fredholm, Q ← compact

Take finite dim subspace $W_1 \subset H_1$ s.t. $W_1 + \text{image}(L) = H_1$

Let $W_0 = L^{-1}(W_1)$. $\quad W_1 \subset H_1 \quad W_0 \subset H_0$

Consider $\widetilde{SW}_{apr} = L + pr_{W_1} \circ Q : W_0 \to W_1$

$\widetilde{SW}^{-1}(0) \cap S_{R+1}(H_0)$
$\cup$

Note, since $\widetilde{SW}^{-1}(0)$ is compact. $\exists R \gg 0$ s.t. $\widetilde{SW}^{-1}(0) \subset B_R(H_0)$

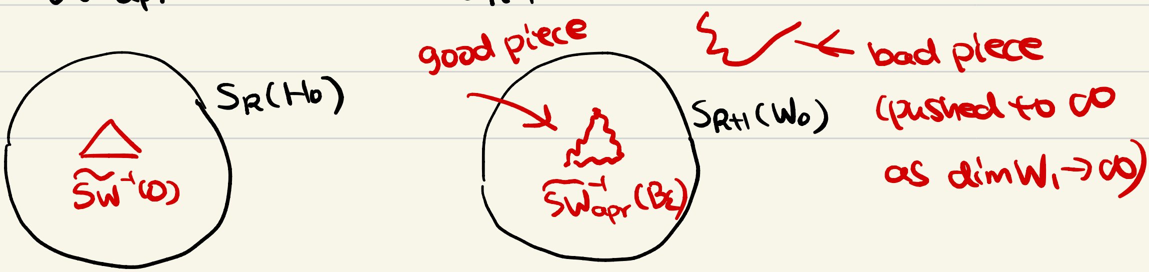
Here $B_R(V) = \{x \in V \mid |x| \leq R\}$ $\qquad S^V = V \cup \{\infty\}$

$S_R(V) = \{x \in V \mid |x| = R\}$.

Using ⊛, one can prove :

Proposition : $\exists \varepsilon$ small and W_1 large s.t.

$\widetilde{SW}_{apr}^{-1}(B_\varepsilon(W_1)) \cap S_{R+1}(W_0) = \emptyset$



$$\underbrace{L + pr_{W_1}\circ Q}_{\widetilde{SW}_{apr}}$$

$\widetilde{SW}_{apr} : W_0 \longrightarrow W_1$ Pin(2)-equivariant.

$$\widetilde{SW}^{-1}(B_\varepsilon(W_1)) \cap S_{R+1}(W_0) = \emptyset$$

This induces a Pin(2)-equivariant map:

$$\widetilde{SW}^{+}_{apr} : S^{W_0} \cong B_{R+1}(W_0)/S_{R+1}(W_0) \to W_1 / W_1 \setminus \mathring{B}_\varepsilon(W_1) \cong S^{W_1}$$

Consider the following representations of Pin(2)

$\mathbb{H}$: 4-dim, Pin(2) acts as left multiplication

$\tilde{\mathbb{R}}$: 1-dim, S^1 acts trivially, j acts reflection.

suppose $Q_X = 2p(-E_8) \oplus q\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $p \geq 0$ $\mathrm{ind}_{\mathbb{C}}(\not{D}_{A_0}) = \frac{-\sigma}{8}$

Then $\mathrm{ind}(L) = \mathrm{ind}(\not{D}_{A_0} \oplus (d^*, d^+))$

$$W_0 - W_1 = p\mathbb{H} - q\tilde{\mathbb{R}} \qquad W_0 = L^{-1}(W_1)$$

This implies that $W_0 \cong (m+p)\mathbb{H} + n\tilde{\mathbb{R}}$ as Pin(2) rep.

$W_1 \cong m\mathbb{H} + (n+q)\tilde{\mathbb{R}}$

So $\underbrace{\widetilde{SW}^{+}_{apr}}_{f}$ is an equivariant map that fits into:

$$\begin{tikzcd} S^{4l-1} \arrow[r, "f"] & S^{l\tilde{\mathbb{R}}} \\ & S^0 \arrow[ul] \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} S^{(m+p)\mathbb{H}+n\tilde{\mathbb{R}}} \arrow[rr, "f"] & & S^{m\mathbb{H}+(n+q)\tilde{\mathbb{R}}} \\ & S^0 = \{0,\infty\} \arrow[ul] \arrow[ur] \arrow[u, phantom, "\circledast"] & \end{tikzcd}$$

Using equivariant K-theory, Furuta proved:

$\exists f$ s.t. $\circledast$ commutes $\Rightarrow q \geq 2p+1$ or $(p,q) = (0,0)$

So if $Q_X \cong 2p(-E_8) \oplus q\begin{pmatrix}0&1\\1&0\end{pmatrix}$, then $q \geq 2p+1$ or $(p,q) = (0,0)$

Using equivariant stable homotopy theory, we proved:

Theorem (Hopkins-L.-Shi-Xu) $\exists f$ s.t. ④ commutes iff

$$q \geq 2p + \begin{cases} 2 & p \equiv 1,2,5,6 \mod 8 \\ 3 & p \equiv 3,4,7 \mod 8 \\ 4 & p \equiv 0 \mod 8 \end{cases} \quad \text{or } (p,q) = (0,0),(0,1),(1,3)$$

This implies an improved bound

$$b_2(X) \geq \frac{10}{8}|\sigma(X)| + c(\sigma) \quad \text{where } c(\sigma) = 4,6,8.$$

This seems to be the limit of this approach.

- The Bauer-Furuta invariant.

Given pointed $Pin(2)$-spaces A, B, we consider the equivariant stable homotopy group.

$$\{A,B\}^{Pin(2)} := \varinjlim_{m,n,k\to\infty} [S^{m\mathbb{H}+n\tilde{\mathbb{R}}+k\mathbb{R}} \wedge A, S^{m\mathbb{H}+n\tilde{\mathbb{R}}+k\mathbb{R}} \wedge B]^{Pin(2)}$$

(Recall: $A \wedge A' := A \times A' / (* \times A') \cup (A \times *)$)

Then $[SW^+_{apr}(X)] \in \{S^{\frac{-\sigma(X)}{16}\mathbb{H}}, S^{b_2^+(X)\tilde{\mathbb{R}}}\}^{Pin(2)}$

It turns out this element is a smooth invariant of X

We denoted by $BF^{Pin(2)}(X,\mathfrak{s})$

For a general spin^c structure $\mathfrak{s}$, we have

$$BF^{S^1}(X,\mathfrak{s}) \in \{S^{\frac{c_1(\mathfrak{s})^2-\sigma(X)}{8}\mathbb{C}}, S^{b_2^+(X)\mathbb{R}}\}^{S^1}$$

must be spin

Note $BF^{Pin(2)}(X,\mathfrak{s}) \Longleftrightarrow BF^{S^1}(X,\mathfrak{s}) \Rightarrow BF^{\{e\}}(X,\mathfrak{s})$

$d(\mathfrak{s})=0$

$$\Updownarrow$$

$SW(X,\mathfrak{s}) \in \mathbb{Z}$

Furthermore, it contains more information.

E.g. $BF^{\{e\}}$ detect exotic $K3\#K3$. (While SW don't).