Extremal Kähler metrics and perturbation problems

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Introduction

These sets of notes were produced for a lecture series the author gave in December 2021 at the invitation of Professor Futaki, hosted through Zoom by Tsinghua University. The goal is to explain two results by the author and collaborators, and put these in the wider context of perturbation problems in Kähler geometry. The first result, which is joint work with Cristiano Spotti, constructs extremal metrics on certain destabilising test configurations. The second, which is joint with Ruadhaí Dervan, constructs extremal metrics on the blowup of a manifold in a point, and gives precise conditions for when this can be achieved.

The series has three lectures. The first is an introductory lecture on the search for canonical metrics in Kähler geometry, with special emphasis on the linear theory crucial to perturbation problems. The second lecture concerns the result with Spotti. The third and final lecture concerns the result with Dervan.

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LECTURE 1

Perturbation problems for extremal Kähler metrics

The goal of the first lecture is to explain some background on Kähler geometry, focusing on constant scalar curvature Kähler (cscK) metrics and extremal Kähler metrics, and perturbation problems for such metrics. A lot of the material follows the book [**41**] by Gábor Székelyhidi.

1. The cscK problem

Let X be a complex manifold. A Riemannian metric g is *Hermitian* if

$$J^* g = g,$$

which means that

$$g(J(\cdot), J(\cdot)) = g(\cdot, \cdot).$$

I.e., multiplication by i is an isometry on each tangent space.

If g is Hermitian, then

$$\omega(\cdot, \cdot) = g(J(\cdot), \cdot)$$

is a 2-form, as $J^2 = -\mathrm{Id}$:

$$\omega(u, v) = g(Ju, v) = J^* g(Ju, v) = g(J^2 u, Jv) = -g(Jv, u) = -\omega(v, u).$$

Note that ω and g determine one another. So we may say Kähler metric ω when we really mean Kähler form ω, etc.

The metric g is *Kähler* if

$$d\omega = 0. \tag{1.1}$$

This means that X simultaneously has the structures of a complex manifold, Riemannian manifold and symplectic manifold, in a compatible way. There are many other important equivalent conditions to the Kähler condition. For example, g is Kähler if and only if for every point in X, there exists "holomorphic normal coordinates", i.e. coordinates $(z^1, \dots, z^n)$ such that

$$\omega = i \sum_{j,\bar{k}} g_{j\bar{k}} dz^j \wedge d\bar{z}^k,$$

where

$$g_{j,\bar{k}} = \delta_{j,k} + O(|z|^2).$$

Note that one can always achieve $g_{j,\bar{k}} = \delta_{j,k} + O(|z|)$, as we can choose coordinates to make the metric the Euclidean one at a given point. The

Kähler condition is precisely what is needed to be able to choose coordinates in which the $O(|z|)$ terms cancel.

By Equation (1.1), if g is Kähler, there is then an associated class

$$\Omega = [\omega] \in H^2(X, \mathbb{R}),$$

the Kähler class of ω. By the $i\partial\bar{\partial}$ Lemma, if $\omega' \in \Omega$ is another Kähler form in the same class, then there exists $\phi : X \to \mathbb{R}$ such that

$$\omega' = \omega + i\partial\bar{\partial}\phi =: \omega_\phi.$$

The set

$$\{\phi : \omega_\phi \text{ is a Kähler form}\} \subseteq_{\text{open}} C^\infty(X)$$

is an open subset in the set of smooth functions on X that parametrises Kähler metrics in the class Ω. In particular, in any Kähler class, there is an infinite dimensional set of Kähler metrics. The following question is therefore very natural:

Question: Is there a canonical representative $\omega_\phi \in \Omega$?

To begin to answer this, we first need to ask ourselves: what is a good notion of canonical metric?

$\dim_{\mathbb{C}} X = 1$: The uniformisation theorem gives a unique metric of constant curvature. This gives a good canonical choice in complex dimension 1.

$\dim_{\mathbb{C}} X > 1$: Should also be some curvature property, but there are many curvature notions, leading to different notions of canonical metrics!

Ricci curvature: In the Kähler case, the Ricci curvature Ric_g also satisfies

$$J^* \mathrm{Ric}_g = \mathrm{Ric}_g .$$

So we get a 2-form $\rho_g = \rho_\omega$, the *Ricci form.*

Local expression: If

$$\omega = i \sum_{j,\bar{k}} g_{j\bar{k}} dz^j \wedge d\bar{z}^k,$$

then

$$\begin{aligned}\rho_\omega &= -i\partial\bar{\partial}\big(\log\det(g_{p\bar{q}})\big)\\ &= -i\sum_{j,\bar{k}} \frac{\partial}{\partial z^j}\frac{\partial}{\partial \bar{z}^k}\big(\log\det(g_{p\bar{q}})\big) dz^j \wedge d\bar{z}^k.\end{aligned}$$

Note that this shows that ρ_ω is closed.

Moreover, if $\rho_{\tilde{\omega}}$ is the Ricci form of another Kähler metric (not necessarily in the same class!) then

$$\rho_\omega - \rho_{\tilde{\omega}} = i\partial\bar{\partial}\Big(\log\frac{\det(\tilde{g}_{p\bar{q}})}{\det(g_{p\bar{q}})}\Big).$$

We have that

$$\frac{\det(\tilde{g}_{p\bar{q}})}{\det(g_{p\bar{q}})} = \frac{\tilde{\omega}^n}{\omega^n},$$

which is a **globally** defined function (here the notation $\frac{\tilde{\omega}^n}{\omega^n}$ means that this is the unique function f such that $\tilde{\omega}^n = f\omega^n$). So

$$[\rho_\omega] = [\rho_{\tilde{\omega}}].$$

This class is

$$2\pi c_1(X) = -2\pi c_1(K_X),$$

since we can interpret the Ricci curvature as the (negative) of the induced curvature on K_X.

Returning to the canonical metric question, we could then ask for ω to be a:

Kähler-Einstein metric:

$$\lambda\omega = \rho_\omega,$$

for some $\lambda \in \mathbb{R}$. But then $\lambda\Omega = 2\pi c_1(X)$. So $c_1(X)$ is either trivial (when $\lambda = 0$) or has a definite sign, and apart from the Calabi-Yau case $c_1(X) = 0$, the class Ω is then even pre-determined up to scale.

So in order to get a condition which makes sense on *any* Kähler manifold and any class, we need to relax the curvature condition.

Scalar curvature: Can contract ρ_ω with the metric to get the scalar curvature

$$\begin{aligned} S(\omega) &= \Lambda_\omega(\rho_\omega) \\ &= -\sum_{j,\bar{k}} g^{j\bar{k}} \frac{\partial}{\partial z^j} \frac{\partial}{\partial \bar{z}^k} \big(\log \det(g_{p\bar{q}}) \big), \end{aligned}$$

locally.

We then seek a *constant scalar curvature (cscK)* metric in Ω, i.e. we wish to solve

$$S(\omega_\phi) = \text{constant}.$$

This question makes sense in any Kähler class. So the initial canonical metric question, for us, in higher dimensions becomes: **Given** (X, Ω) **does there exist an** $\omega \in \Omega$ **with constant scalar curvature?**

Note that the constant is predetermined by the class: The operator $\Lambda_\omega(\alpha)$, which locally is $g^{i\bar{j}}\alpha_{i\bar{j}}$, satisfies

$$\Lambda_\omega(\alpha)\omega^n = n\alpha \wedge \omega^{n-1}.$$

So

$$S(\omega)\omega^n = n\rho_\omega \wedge \omega^{n-1},$$

and therefore

$$\begin{aligned} \int_X S(\omega)\omega^n &= n \int_X \rho_\omega \wedge \omega^{n-1} \\ &= 2\pi n c_1(X) \cdot \Omega^{n-1}. \end{aligned}$$

So if $S(\omega)$ is constant, then it equals

$$\hat{S}_\Omega = \frac{2\pi n c_1(X) \cdot \Omega^{n-1}}{\Omega^n}.$$

2. Obstructions to cscK metrics

2.1. Futaki's obstruction to cscK metrics. The aim of this talk is to discuss perturbation problems for the cscK equation. As a way of introducing some of the computations involved in this, we first show that there is an obstruction to the existence of cscK metrics due to Prof. Futaki, coming from holomorphic vector fields on X. This will also help motivative the study of extremal Kähler metrics, a more general type of canonical metrics.

DEFINITION 1.1. A function h on X is a *holomorphy potential (wrt ω)* if $\nabla^{1,0}_{\omega} h$ is a holomorphic vector field, i.e.

$$\mathcal{D}_{\omega}(h) := \bar{\partial}(\nabla^{1,0} h) = 0.$$

It is a result of LeBrun-Simanca that the holomorphic vector fields that admit a holomorphy potential are precisely the ones with a zero somewhere. So having a holomorphy potential does not depend on the Kähler metric chosen. We will now show something weaker, namely that having a potential is independent of the metric chosen in a fixed class.

LEMMA 1.2. *If ν is a holomorphic vector field on X with potential h with respect to ω, then $h + \nu(\phi)$ is a holomorphy potential with respect to ω_{ϕ}.*

PROOF. Locally, we have $\nu = \sum_j \nu^j \frac{\partial}{\partial z^j}$, with $\frac{\partial}{\partial \bar{z}^k} \nu^j = 0$ for all j, k, since ν is holomorphic. Moreover, since h is a holomorphy potential with respect to ω, we have

$$\nu^j = \sum_k g^{j\bar{k}} \frac{\partial h}{\partial \bar{z}^k}.$$

This is because $g(\nu, \frac{\partial}{\partial \bar{z}^k}) = g(\nabla h, \frac{\partial}{\partial \bar{z}^k}) = dh(\frac{\partial}{\partial \bar{z}^k})$.

Now, if $g_{\phi, j\bar{p}}$ denotes the components of g_{ϕ},

$$\begin{aligned}
\sum_j g_{\phi, j\bar{p}} \nu^j &= \sum_j (g_{j\bar{p}} + \phi_{j\bar{p}}) \nu^j \\
&= \sum_j g_{j\bar{p}} (\sum_k g^{j\bar{k}} \partial_{\bar{k}} h) + \phi_{j\bar{p}} \nu^j \\
&= \sum_k \delta_{p,k} \partial_{\bar{k}} h + \sum_j \phi_{j\bar{p}} \nu^j \\
&= \partial_{\bar{p}}(h) + \partial_{\bar{p}} \big(\sum_j \phi_j \nu^j \big) \\
&= \partial_{\bar{p}}(h + \nu(\phi)).
\end{aligned}$$

In the second to last line, we used that the ν^j are holomorphic.

Applying the inverse to $g_{\phi, j\bar{p}}$ we then get that

$$\nu^j = \sum_p g_{\phi}^{j\bar{p}} \partial_{\bar{p}}(h + \nu(\phi)),$$

and so

$$\nu = \nabla^{1,0}_{\omega_\phi}(h + \nu(\phi)),$$

as required. □

DEFINITION 1.3. Let $\nu \in \mathfrak{h}$, the space of holomorphic vector fields with a potential. The *Futaki invariant* is the functional

$$F_\omega : \mathfrak{h} \to \mathbb{C}$$

defined by

$$F_\omega(v) = \int_X h_\omega(S(\omega) - \hat{S}_\Omega)\omega^n,$$

where h_ω is a holomorphy potential for ν with respect to ω.

A-priori, this depends on ω. Futaki showed that this is not the case.

THEOREM 1.4 (Futaki, [**23**]).

$$F_\omega = F_\Omega$$

is independent of the metric ω *chosen in* Ω.

COROLLARY 1.5. *If there is* $\tilde{\omega} \in \Omega$ *which is cscK, then* $F_\Omega = 0$.

So the Futaki invariant gives an obstruction to the existence of a cscK metric in a given class. E.g., in any class on $X = \mathrm{Bl}_p\,\mathbb{P}^2$ one can show that the Futaki invariant is non-zero, and so there is no cscK metric in any class on this manifold (there are however extremal metrics, a more general form of canonical metrics introduced by Calabi, that we are discussing a bit later).
Aside: For *toric* Fano manifolds, with $\Omega = c_1(K_X^*)$, this is the only obstruction. This is the "barycenter 0" condition.

PROOF OF THEOREM 1.4. We want to show that

$$F_{\omega_\phi}(\nu) = \int_X h_\phi(S(\omega_\phi) - \hat{S}_\Omega)\omega_\phi^n$$

equals $F_\omega(\nu)$, where we are using the shorthand h_ϕ for the changed potential, given by Lemma 1.2. It suffices to show that for any ω and for all $\phi \in C^\infty(X)$,

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\big(F_{\omega_{t\phi}}(\nu)\big) = 0.$$

From Lemma 1.2, we know $h_{t\phi} = h + \nu(t\phi)$, and so

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\big(h_{t\phi}\big) = \nu(\phi).$$

Also,

$$\begin{aligned}\big(\omega_{t\phi}\big)^n &= \big(\omega + ti\partial\bar{\partial}\phi\big)^n \\ &= \omega^n + nti\partial\bar{\partial}\phi \wedge \omega^{n-1} + \cdots,\end{aligned}$$

and so

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\left(\left(\omega_{t\phi}\right)^n\right) = ni\partial\bar{\partial}\phi \wedge \omega^{n-1}$$
$$= \Lambda_\omega(i\partial\bar{\partial}\phi)\omega^n$$
$$= \Delta_\omega(\phi)\omega^n,$$

the Laplacian applied to ϕ.

We also need to know the linearisation of the scalar curvature. Since

$$S(\omega_{t\phi})\omega_{t\phi}^n = n\rho_{t\phi} \wedge \omega_{t\phi}^{n-1}, \tag{2.1}$$

this can be computed from the linearisations of $\omega_{t\phi}^k$, $k = n, n-1$, and from the linearisation of the Ricci curvature. To see the change in the linearisation of the Ricci curvature, we use that

$$\rho_{\omega_{t\phi}} - \rho_\omega = -i\partial\bar{\partial}\big(\log \frac{\omega_{t\phi}^n}{\omega^n}\big),$$

and so

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\left(\rho_{t\phi}\right) = -i\partial\bar{\partial}\left(\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\big(\log \frac{\omega_{t\phi}^n}{\omega^n}\big)\right)$$
$$= -i\partial\bar{\partial}\left(\frac{\omega^n}{\omega_{t\phi}^n}\Big|_{t=0}\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\big(\frac{\omega_{t\phi}^n}{\omega^n}\big)\right)$$
$$= -i\partial\bar{\partial}\big(\Delta(\phi)\big).$$

Continuing using the identity (2.1),

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\big(S(\omega_{t\phi})\big)\omega^n = -ni\partial\bar{\partial}\left(\Delta(\phi)\right) \wedge \omega^{n-1} + n(n-1)\rho \wedge i\partial\bar{\partial}(\phi) \wedge \omega^{n-2}$$
$$- S(\omega)\Delta_\omega(\phi)\omega^n$$
$$= \big(-\Delta^2(\phi) + S(\omega)\Delta(\phi) - \langle \rho, i\partial\bar{\partial}\phi\rangle - S(\omega)\Delta_\omega(\phi)\big)\,\omega^n,$$
$$= -\left(\Delta^2(\phi) + \langle \rho, i\partial\bar{\partial}\phi\rangle\right)\omega^n,$$

where we have used $n(n-1)\alpha \wedge \beta \wedge \omega^{n-2} = \left(\Lambda_\omega(\alpha)\Lambda_\omega(\beta) - \langle \alpha, \beta\rangle\right)\omega^n$.

Finally one obtains

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\left(S(\omega_{t\phi})\right) = -\mathcal{D}_\omega^*\mathcal{D}_\omega(\phi) + \langle \nabla^{1,0}\phi, \nabla^{1,0}S(\omega)\rangle_\omega, \tag{2.2}$$

where we recall $\mathcal{D}_\omega(\phi) = \bar{\partial}(\nabla_\omega^{1,0}\phi)$. Using that $S(\omega)$ is real, we can conjugate and also obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\left(S(\omega_{t\phi})\right) = -\overline{\mathcal{D}_\omega^*\mathcal{D}_\omega(\phi)} + \langle \nabla^{1,0}S(\omega), \nabla^{1,0}\phi\rangle_\omega. \tag{2.3}$$

Remark: One obtains an equality involving the squared Laplacian from (2.1) and our computations, and one then needs to know something about how $\mathcal{D}^*\mathcal{D}$ compares to Δ^2 to obtain this identity. The formula is

$$\mathcal{D}^*\mathcal{D} = \Delta^2(\phi) + \langle \rho, i\partial\bar{\partial}\phi \rangle + \langle \nabla^{1,0}\phi, \nabla^{1,0}S(\omega)\rangle_\omega.$$

With all of this in place, we can now compute the derivative of the Futaki invariant in the direction of ϕ:

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\left(F_{\omega_{t\phi}}(\nu)\right) = \int_X \Big(v(\phi)(S(\omega) - \hat{S}_\Omega) + h(S(\omega) - \hat{S}_\Omega)\Delta(\phi) - h\overline{\mathcal{D}^*_\omega \mathcal{D}_\omega(\phi)} + h\langle \nabla^{1,0}S(\omega), \nabla^{1,0}\phi\rangle \Big)\omega^n,$$

Note that because of the Kähler identity $[\Lambda_\omega, \partial] = i\bar{\partial}^*$, we have

$$\Delta = \Lambda_\omega(i\partial\bar{\partial}) = i[\Lambda_\omega, \partial]\bar{\partial} = -\bar{\partial}^*\bar{\partial},$$

and so we have the identity

$$\int_X f\Delta(\phi)\omega^n = -\int_X \langle \nabla^{1,0}f, \nabla^{1,0}\phi\rangle\omega^n.$$

Applying this to $f = h(S(\omega) - \hat{S}_\Omega)$ we get that

$$\begin{aligned}\int_X h(S(\omega) - \hat{S}_\Omega)\Delta(\phi)\omega^n &= -\int_X \langle \nabla^{1,0}\big(h(S(\omega) - \hat{S})\big), \nabla^{1,0}\phi\rangle\omega^n \\ &= -\int_X h\langle \nabla^{1,0}\big(S(\omega)\big), \nabla^{1,0}\phi\rangle\omega^n \\ &\quad -\int_X (S(\omega) - \hat{S})\langle \nabla^{1,0}h, \nabla^{1,0}\phi\rangle\omega^n.\end{aligned}$$

Moreover, $\langle \nabla^{1,0}h, \nabla^{1,0}\phi\rangle = \langle \nu, \nabla\phi\rangle = d\phi(\nu) = \nu(\phi)$. Putting this back in the derivative of the Futaki invariant, all we are left with is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\left(F_{\omega_{t\phi}}(\nu)\right) &= -\int_X h\overline{\mathcal{D}^*_\omega \mathcal{D}_\omega(\phi)}\omega^n \\ &= -\int_X \mathcal{D}^*_\omega \mathcal{D}_\omega(h)\phi\omega^n \\ &= 0,\end{aligned}$$

since h is a holomorphy potential, and therefore is in the kernel of $\mathcal{D}_\omega$. □

2.2. K-stability and the YTD conjecture. There are more obstructions to the existence of cscK/extremal metrics, coming from degenerating X to other manifolds, or even singular varieties. K-stability is such a notion. It involves a class of a degenerations, called test configurations, and a numerical invariant, the Donaldson–Futaki invariant, that one associates to each test configuration. K-stability then asks for this to have a particular sign.

The class of degenerations are defined as follows. Let (X, L) be a polarised manifold or variety. A *test configuration* $(\mathcal{X}, \mathcal{L})$ of exponent r is a normal polarised variety with a map $\pi : \mathcal{X} \to \mathbb{C}$ such that

- $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{C}$ is a flat family;
- $(\mathcal{X}, \mathcal{L})$ admits a $\mathbb{C}^*$-action, such that π is equivariant with respect to the standard action on $\mathbb{C}$;
- all non-zero fibres $(\mathcal{X}_t, \mathcal{L}_t)$ are isomorphic to (X, L^r).

From a test configuration, over $\mathbb{C}$ as above, one can define a compactified test configuration over $\mathbb{P}^1$, simply by gluing with $X \times \mathbb{C}$, using that away from the fibre over 0 (the *central fibre*), $\mathcal{X}$ is isomorphic to $X \times \mathbb{C}^*$. Sometimes we will think of test configurations in this way.

Now, we will associate a number to a test configuration. Originally, these were defined in terms of asymptotic expansions of the dimension of certain vector spaces and weights of actions on these vector spaces. However, they have since been shown by Odaka and Wang to be given by an intersection number on the total space of the compactified test configuration ([**31, 44**]), and it is this formula we give.

So, let $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{P}^1$ be a test configuration over $\mathbb{P}^1$, of exponent r. Let $K_{\mathcal{X}/\mathbb{P}^1} = K_{\mathcal{X}} - \pi^* K_{\mathbb{P}^1}$ denote the relative canonical bundle, and let $\mu(X, L) = \frac{-K_X . L^{n-1}}{L^n}$ denote the slope of (X, L). Then the Donaldson–Futaki invariant of $(\mathcal{X}, \mathcal{L})$ is given by

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = \frac{n}{n+1} \mu(X, L^r) \mathcal{L}^{n+1} + K_{\mathcal{X}/\mathbb{P}^1} . \mathcal{L}^n.$$

In the case when $(\mathcal{X}, \mathcal{L})$ comes from a *product test configuration* $X \times \mathbb{C}$ with an action on the X-component generated by a holomorphic vector field on X, one recovers the classical Futaki invariant of the vector field.

DEFINITION 1.6. The polarised normal variety (X, L) is

- *K-semistable* if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations;
- *K-stable* if further $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$ if and only if $(\mathcal{X}, \mathcal{L})$ is a trivial test configuration;
- *K-stable* if further $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$ if and only if $(\mathcal{X}, \mathcal{L})$ is a product test configuration.

A central conjecture in the field is the Yau–Tian–Donaldson (YTD) conjecture:

CONJECTURE 1.7 ([**45, 43, 20**]). *A polarised projective manifold (X, L) admits a cscK metric in $c_1(L)$ if and only if it is K-polystable.*

We remark that in general one expects that a stronger stability notion, such as a uniform K-stability notion, is needed for the conjecture to hold. One should view finding the correct stability notion as part of the problem. On the other hand, for example for Fano manifolds with the anticanonical polarisation, the existence of a Kähler–Einstein metric is in fact equivalent to the K-stability – no stronger stability notion is needed ([**12, 13, 14**]). There are also variants of the above for non-projective manifolds/polarisations, as well an extension to the extremal setting that we discuss below, where the relevant stability notion is *relative* K-stability ([**38**]).

3. Extremal Kähler metrics

There is an even more general notion of canonical metric, due to Calabi ([**9, 10**]). It generalises cscK metrics in the case when the reduced automorphism group is non-zero, which is precisely when the Futaki invariant provides an obstruction to the existence of cscK metrics.

Extremal Kähler metrics are defined as the critical points of the energy functional associated to the scalar curvature operator. In other words, as critical points of the functional

$$\mathcal{K}_\omega \to \mathbb{R}$$

given by

$$\phi \mapsto \int_X S(\omega_\phi)^2 \omega_\phi^n.$$

When X is compact (which we always assume), this is equivalent to

$$\mathcal{D}_{\omega_\phi}(S(\omega_\phi)) = 0,$$

where

$$\mathcal{D}_{\omega_\phi} = \bar{\partial}(\nabla^{1,0}_{\omega_\phi}(f))$$

is the operator we saw in the computation of the linearisation of the scalar curvature operator earlier (it is a straightforward exercise given the linearisation calculations we have done to verify that this indeed is the case).

Definition 1.8 (Calabi). A Kähler metric ω on a compact Kähler manifold is *extremal* if

$$\mathcal{D}_\omega(S(\omega)) = 0.$$

This clearly generalises cscK metrics, since if $S(\omega)$ is constant, then the gradient is zero. Moreover, the Futaki invariant captures precisely when an extremal metric is cscK.

Proposition 1.9. *Suppose $\omega \in \Omega$ is an extremal Kähler metric. Then ω is cscK if and only the Futaki invariant F_Ω vanishes.*

Proof. We already know from Corollary 1.5 that the Futaki invariant vanishes if ω is cscK. Conversely, suppose that ω is extremal, but not cscK. Then we can pick $h = S(\omega) - \hat{S}$ as the holomorphy potential for the extremal vector field

$$\nu = \nabla^{1,0}_\omega(S(\omega))$$

in the Futaki invariant to get that

$$F_\Omega(\nu) = \int_X \left(S(\omega) - \hat{S}\right)^2 \omega^n > 0.$$

So the Futaki invariant does not vanish in this case. □

Since the scalar curvature map is a fourth order operator, the extremal equation

$$\mathcal{D}_\omega(S(\omega)) = 0$$

is order six. However, we can also view the equation as

$$S(\omega) \in \overline{\mathfrak{h}}_{\omega},$$

where $\overline{\mathfrak{h}}_{\omega}$ is the finite dimensional space of potentials for real holomorphic vector fields on X. In this way we can view the equation as a fourth order equation.

Note that the space $\overline{\mathfrak{h}}_{\omega}$ depends on ω. Indeed, given a holomorphic vector field with zeros, we know the potential for it will change with the Kähler metric (Lemma 1.2). We are thus trying to hit a moving target. When doing analysis it is convenient to work with fixed spaces of functions, and we will now look into how we can do this for the extremal equation, when phrasing the equation in this way.

First note that while the holomorphic vector fields do not depend on ω, the ones with a real potential, i.e. the real holomorphic vector fields, do. However, if one fixes a maximal torus T of the reduced automorphism group of X and only works with torus-invariant metrics and potentials, then this stops being the case. Thus we will always assume from now (without mention) that we are have fixed a maximal torus and are working with torus-invariant metrics and functions.

Having done this, we are assured that when we use Lemma 1.2, a real potential remains real after changing the metric by a torus-invariant function. If

$$h_1, \dots, h_r$$

form a basis for the real holomorphy potentials on X with respect to ω, we then have that

$$h_1 + \frac{1}{2}\langle \nabla h_1, \nabla \phi \rangle, \dots, h_r + \frac{1}{2}\langle \nabla h_r, \nabla \phi \rangle$$

form a basis for the real holomorphy potentials on X with respect to ω_{ϕ}. Thus to solve the extremal equation on X, we want to find a (torus-invariant) function $\phi : X \to \mathbb{R}$ and a holomorphy potential h such that

$$S(\omega_{\phi}) = h + \frac{1}{2}\langle \nabla h, \nabla \phi \rangle.$$

In this way, we can view the equation as an equation between fixed spaces. If $\overline{\mathfrak{h}}$ denotes the space of potentials for real holomorphic vector fields with respect to ω, we seek the root of the map $C^{k+4,\alpha} \times \overline{\mathfrak{h}} \to C^{k,\alpha}$ given by

$$(\phi, h) \mapsto S(\omega_{\phi}) - h - \frac{1}{2}\langle \nabla h, \nabla \phi \rangle.$$

In fact, one can show that the h we are trying to hit is predetermined, once the maximal torus is fixed (this is due to Futaki–Mabuchi [**24**]).

4. Perturbation problems

We now come to the main point of the talk, where we discuss the overall strategy involved in perturbation problems for the extremal equation.

These problems are of the following type. One starts with a pair (X, Ω) such that Ω admits a cscK or extremal metric. Then one perturbs the problem a bit, changing the geometric setup to another pair $(\tilde{X}, \tilde{\Omega})$, say, that is "close" to (X, Ω). Typically this involves some parameter, say ε, so that we have a family $(\tilde{X}_\varepsilon, \tilde{\Omega}_\varepsilon)$ and in some sense as $\varepsilon \to 0$, we get closer and closer to our original (X, Ω), where we know there is a cscK or extremal metric. Often $\tilde{X}_\varepsilon$ does not depend on $\varepsilon > 0$, but may be different from X. See e.g. the theorems of Arezzo–Pacard, and their generalisations, discussed below, on the existence of cscK metrics on blowups of cscK manifolds.

Sample problems:

- LeBrun–Simanca openness:
$$(\tilde{X}_\varepsilon, \tilde{\Omega}_\varepsilon) = (X, \Omega + \varepsilon A).$$
- Arezzo–Pacard type blow up:
$$(\tilde{X}_\varepsilon, \tilde{\Omega}_\varepsilon) = (\mathrm{Bl}_p X, \pi^*\Omega - \varepsilon[E]).$$
- Fibrations (Fine, Hong, ...):
$$(\tilde{X}_\varepsilon, \tilde{\Omega}_\varepsilon) = (X, \varepsilon\Omega + \pi^*\Omega_B),$$
where X is the total space of a fibration $\pi : X \to B$ where the fibres X_b admit cscK metrics in $\Omega_{|X_b}$, and Ω_B is a Kähler class on B.

We will discuss the first problem today and the two latter in the next two lectures.

Back to the general picture: One then uses the extremal metric on X in Ω to construct an approximately extremal metric $\omega_\varepsilon \in \tilde{\Omega}_\varepsilon$. The goal is then to show via some contraction mapping principle that this can be perturbed to a genuine solution of the extremal equation, at least when the parameter ε is very close to 0.

This often relies on a quantitative inverse/implicit function theorem. We know from the inverse function theorem that if the linearisation of an operator $\mathcal{N}$ is invertible, then we can hit everything near $\mathcal{N}(0)$. However, in our ε-dependent situation we have a one parameter family $\mathcal{N}_\varepsilon$ of operators and while $\mathcal{N}_\varepsilon(0) \to 0$ as $\varepsilon \to 0$, it may be that the neighbourhood of $\mathcal{N}_\varepsilon(0)$ that we can hit also shrinks with ε. So in order to guarantee that we can solve the cscK equation when ε is sufficiently small, we need to show that the approximate solutions get better and better at a sufficiently fast rate. This will not really be an issue in the first of the problems mentioned above, but will feature on both of the latter two.

This rate is determined by bounds on the inverse of the linearised operator. Recall from equation (2.2) that the linearisation of the scalar curvature operator is
$$-L_\omega + \frac{1}{2}\langle \nabla S(\omega), \nabla(\cdot)\rangle.$$

In particular, if $S(\omega)$ is constant, this linearised operator is the Lichnerowicz operator

$$L_\omega = -\mathcal{D}^*_\omega \mathcal{D}_\omega(\cdot),$$

and so the mapping properties of this operator is typically key in these problems. Note that if X does not admit holomorphic vector fields with zeros, the cokernel of this operator is the constants, but if X does admit holomorphic vector fields, then this cokernel increases. This usually complicates these type of problems when there are holomorphic vector fields present.

5. The LeBrun–Simanca openness theorem, a sample problem

We will now give discuss a problem of the above type in more detail. We outline a proof. The proof is a little bit different from the original proof, and is more reminiscent of the style of proof involved in the other perturbation problems mentioned.

The theorem we wish to prove is the following, originally due to LeBrun–Simanca.

THEOREM 1.10 ([**29**, Theorem A]). *The set of Kähler classes that admit an extremal metric is an open subset of the Kähler cone.*

In other words, we wish to show that if Ω is a Kähler class that admits extremal metrics, then there exists an open subset $U \subset H^{1,1}(X) \cap H^2(X, \mathbb{R})$ about the origin such that for all $A \in U$, there exists an extremal Kähler metric in the class $\Omega + A$.

5.1. Case 1: no holomorphic vector fields. We start with (X, Ω) admitting a cscK metric ω, and we will assume there are no holomorphic vector fields. Let $A = [\alpha] \in H^{1,1}(X, \mathbb{R})$. Then we can try to solve the cscK equation in the class

$$\Omega + \varepsilon A,$$

which is Kähler for all ε sufficiently small. So we want to solve

$$S(\omega + \varepsilon\alpha + i\partial\bar{\partial}\phi) = c_{\varepsilon,A},$$

where $c_{\varepsilon,A} = \hat{S}_{\Omega+\varepsilon A}$ is a topological constant approaching $\hat{S}_\Omega$ as $\varepsilon \to 0$. This is a perturbation problem of the type we have been discussing where we are not changing the manifold, only the class.

The first step is then to construct a good approximate solution. We are simply going to use $\omega_\varepsilon = \omega + \varepsilon\alpha$. Note that this may not be Kähler for all ε for which $\Omega + \varepsilon A$ is Kähler, but at least it is Kähler for all ε sufficiently close to 0, by the positivity of ω. Moreover, we can certainly ensure $S(\omega_\varepsilon)$ is bounded independently of ε when ε is sufficiently small.

To see that this indeed is going to give better and better approximate solutions, we again use the identity

$$S(\omega_\varepsilon)\omega_\varepsilon^n = n\rho_{\omega_\varepsilon} \wedge \omega_\varepsilon^{n-1}.$$

We have that

$$\begin{aligned}\omega_\varepsilon^n &= (\omega+\varepsilon\alpha)^n \\ &= \omega^n + n\varepsilon\alpha\wedge\omega^{n-1}+\cdots \\ &= \big(1+\varepsilon\Lambda_\omega(\alpha)+\cdots\big)\omega^n\end{aligned}$$

and using the expression established earlier for $\rho_\omega - \rho_{\omega+\varepsilon\alpha}$, we similarly get

$$\begin{aligned}&\rho_{\omega_\varepsilon}\wedge\omega_\varepsilon^{n-1} \\ &= \big(\rho_\omega - \varepsilon i\partial\bar\partial(\Lambda_\omega(\alpha))+\cdots\big)\wedge\big(\omega^{n-1}+(n-1)\varepsilon\alpha\wedge\omega^{n-2}\big) \\ &= \rho_\omega\wedge\omega^{n-1}+\varepsilon\big(-i\partial\bar\partial(\Lambda_\omega(\alpha))\omega^{n-1}+(n-1)\rho_\omega\wedge\alpha\wedge\omega^{n-2}\big)+O(\varepsilon^2).\end{aligned}$$

So

$$\begin{aligned}&S(\omega_\varepsilon)\omega^n \\ =&S(\omega)\omega^n \\ &+\varepsilon\big(-S(\omega_\varepsilon)\Lambda_\omega(\alpha)\omega^n - ni\partial\bar\partial(\Lambda_\omega(\alpha))\omega^{n-1}+n(n-1)\rho_\omega\wedge\alpha\wedge\omega^{n-2}\big) \\ &+O(\varepsilon^2) \\ =&S(\omega)\omega^n \\ &+\varepsilon\big(-S(\omega_\varepsilon)\Lambda_\omega(\alpha)-\Delta(\Lambda_\omega(\alpha))-\langle\rho_\omega,\alpha\rangle+S(\omega)\Lambda_\omega(\alpha)\big)\omega^n \\ &+O(\varepsilon^2).\end{aligned}$$

Note that while $S(\omega_\varepsilon)$ appears on both sides of this equation, the fact that it is bounded allows us to deduce that there is a constant C such that if ε is sufficiently small, then

$$\|S(\omega_\varepsilon)-S(\omega)\|\le C\varepsilon.$$

This is in C^0, but one could apply similar ideas to get bounds for the derivatives too. So ω_ε is an approximate solution to the cscK equation, approaching a cscK metric at $O(\varepsilon)$.

We now wish to perturb this into a genuine solution. That is, we wish to show that we can obtain a zero of the map

$$\mathcal{N}_\varepsilon : C^\infty(X)\times\mathbb{R}\to C^\infty(X)$$

given by

$$(\phi,c)\mapsto S(\omega_\varepsilon+i\partial\bar\partial\phi)-S(\omega)-c.$$

Our previous computation is then saying $\mathcal{N}_\varepsilon(0,0)=O(\varepsilon)$ is an approximate root of this map.

To perturb, we need to look at the linearisation, which we saw was given by

$$P_\varepsilon = -L_{\omega_\varepsilon}(\cdot)+\frac{1}{2}\langle\nabla S(\omega_\varepsilon),\nabla\cdot\rangle.$$

This is a complicated ε-dependent operator. But since ω_ε approaches ω and is approximately cscK, it is well approximated by $P=-L_\omega$, a self-adjoint operator that we know well. This means that

$$\|P_\varepsilon - P\| \le C'\varepsilon$$

in operator norm.

Under our assumption that there are no holomorphic vector fields, P is an isomorphism on $C_0^\infty(X) \times \mathbb{R}$, where the subscript denotes functions of average 0, or more appropriately, it is an isomorphism on the Hölder spaces $C_0^{k+4,\alpha}(X) \times \mathbb{R} \to C^{k,\alpha}(X)$, since we need to work in Banach spaces (elliptic regularity theory will then allow us to show that a solution to the cscK equation is in fact smooth). Since this is an open condition, it follows that P_ε is an isomorphism too, when ε is sufficiently small, and that we even get a bound

$$\frac{1}{C''}\|Q\| \le \|Q_\varepsilon\| \le C''\|Q\|$$

for the inverse Q_ε of P_ε, for some constant C'' independent of ε.

Now we can complete the proof of the existence of the cscK metric: Since the inverse of the linearisation of $\mathcal{N}_\varepsilon$ has a definite bound, the quantitative inverse function theorem implies that there is a neighbourhood of $\mathcal{N}_\varepsilon(0,0)$ of definite size, independently of ε, that can be hit with $\mathcal{N}_\varepsilon$. In particular, since $\mathcal{N}_\varepsilon(0,0) \to 0$ as $\varepsilon \to 0$, there is a $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists $(\phi_\varepsilon, c_\varepsilon)$ such that $\mathcal{N}_\varepsilon(\phi_\varepsilon, c_\varepsilon) = 0$, i.e.

$$S(\omega + \varepsilon\alpha + i\partial\bar{\partial}\phi_\varepsilon) = S(\omega) + c_\varepsilon,$$

a constant.

5.2. Case 2: there are holomorphic vector fields. If X admits holomorphy potentials, then we have a larger cokernel than the constants for $\mathcal{L}_\omega$ and so a larger cokernel for the linearised operator. The cokernel is then holomorphy potentials with respect to ω. If we start with a cscK metric, this gives an obstruction to solving the cscK equation in nearby classes (in fact, this is the Futaki invariant we discussed earlier). However, by incorporating the change in holomorphy potentials with α and ϕ, one can always solve for

$$S(\omega + \varepsilon\alpha + i\partial\bar{\partial}\phi)$$

being a holomorphy potential with respect to $\omega + \varepsilon\alpha + i\partial\bar{\partial}\phi$, i.e. we can find an extremal metric in nearby classes. Restricting to starting with a cscK metric is then unnecessary, and we assume that our initial ω is an extremal metric on X.

We will show that that if one starts with an extremal metric, one can perturb to another extremal metric in nearby classes, but note that if one starts with a cscK metric, it may or may not be that the extremal metric obtained in the perturbed class is cscK. Whether or not this is the case is exactly captured by the Futaki invariant. For example, $X = \mathrm{Bl}_{p_1,p_2,p_3}\mathbb{P}^2$ admits a Kähler-Einstein metric. For this class, i.e. $\Omega = c_1(-K_X)$, the volume of the three exceptional divisors are the same. All nearby Kähler classes admit extremal metrics, but they are cscK if and only if the volume of the exceptional divisors remain the same as it is only when the exceptional divisors have the same volume that the Futaki invariant vanishes.

We now return to the general picture. To be more precise, suppose we start with having an extremal metric ω in some class Ω. If h_ν is the potential for a holomorphic vector field ν on X with respect to ω and h_ν^α is a holomorphy potential for ν with respect to $\omega + \alpha$ (assume α is small enough that this is Kähler), then $h_{\nu,\varepsilon} = h_\nu + \varepsilon\,(h_\nu^\alpha - h_\nu)$ is a holomorphy potential for ν with respect to $\omega_\varepsilon = \omega + \varepsilon\alpha$. If we perturb to $\omega_\varepsilon + i\partial\bar{\partial}\phi$, the new holomorphy potential is

$$h_{\nu,\varepsilon} + \frac{1}{2}\langle \nu, \nabla\phi\rangle.$$

The equation we wish to solve is then

$$S(\omega + \varepsilon\alpha + i\partial\bar{\partial}\phi_\varepsilon) = h_{\nu,\varepsilon} + \frac{1}{2}\langle \nu, \nabla\phi\rangle + c_\varepsilon,$$

to obtain an extremal metric. I.e. we wish to find a zero of the map

$$\Phi : C^{4,\alpha}(X) \times \mathfrak{h} \times \mathbb{R} \to C^{0,\alpha}(X),$$

given by

$$(\phi, \nu, c) \mapsto S(\omega + \varepsilon\alpha + i\partial\bar{\partial}\phi_\varepsilon) - h_{\nu,\varepsilon} - \frac{1}{2}\langle \nu, \nabla\phi\rangle - c.$$

The linearisation of this operator is surjective (because we kill off the cokernel with the $\mathfrak{h} \times \mathbb{R}$ factor.... the term $\frac{1}{2}\langle \nu, \nabla\phi\rangle$ precisely kills off a bad looking term coming from the linearisation of the scalar curvature at a non-cscK metric), and so by following the steps above (rewriting the equation in a form that singles out the leading order term and establishing mapping properties of the linearised operator), we can use a quantitative *implicit* function theorem to obtain the required extremal metric.

Remark 1.11. Technically, we have only showed openness about Ω in the one-parameter family $\Omega + \varepsilon A$ of Kähler classes on X. However, using the finite-dimensionality of the space in which Ω lies (namely $H^{1,1}(X) \cap H^2(X, \mathbb{R})$), it is straightforward to see that we get the required openness statement.

LECTURE 2

Extremal metrics on destabilising test configurations

In this lecture, we will discuss joint work with Cristiano Spotti, where we construct extremal metrics on the total space of certain destabilising test configurations. A simplified version of the main result is the following.

THEOREM 2.1 ([**32**]). *Suppose (X, L) is analytically strictly K-semistable and has discrete automorphism group. On certain test configurations*

$$\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{P}^1,$$

there exists an extremal metric

$$\Omega_k \in c_1(\mathcal{L} + \pi^* \mathcal{O}(k))$$

for all $k \gg 0$.

The actual result is more general, and we will discuss the more refined statement in due course. However, to put the results and techniques in context, we will begin by discussing other constructions of extremal metrics with a similar setup.

1. The general setup

The main result falls in the context of constructing extremal metrics on the total space of fibrations. This has a long history of study, with many different constructions. The general setup is that of a holomorphic submersion (fibration)

$$\pi : (Y, H) \to B,$$

where $H \to Y$ is an ample line bundle on Y. The goal is then to construct Kähler metrics on Y in H (or maybe another polarisation, somehow related to H).

Two of the main avenues of producing extremal Kähler metrics on Y are:

Symmetry: These are explicit constructions using an ansatz in situations with high symmetry. E.g., Calabi's first examples of non-cscK extremal metrics on Hirzebruch surfaces $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(d)) \to \mathbb{P}^1$ ([**9, 10**]), generalisations of these by Hwang–Singer ([**27**]), Apostolov–Calderbank –Gauduchon–Tønnesen-Friedman ([**1**]) (which for example was used by Legendre [**30**] and Jubert ([**28**]) to show that $\mathbb{P}(L_1 \oplus L_2 \oplus L_3) \to C$ is Calabi dream, if C is a genus 0 or 1 curve), ...

Perturbation techniques: These involve working in "adiabatic classes". These are classes of the form $c_1(H+kD)$ for $k \gg 0$, where D is the pullback of an ample line bundle on B. This includes work of Fine [**22**], Hong [**26**], Brönnle ([**8**]), Dervan–S. ([**17, 19**]), and more.

The new construction is of the latter type. The starting point for considering adiabatic classes is the observation that if $\omega_Y \in c_1(H)$ is relatively Kähler and $\omega_B \in c_1(D)$ is Kähler, then the form $\omega_k = \omega_Y + k\omega_B \in c_1(H+kD)$ is Kähler, and its scalar curvature expands as

$$S(\omega_k) = S(\omega_F) + O(k^{-1}),$$

where $S(\omega_F)$ is the scalar curvature of ω_Y restricted to the fibres, the fibrewise scalar curvature. The heurestic reason is that for a product, one would have $S(\omega_k) = S(\omega_F) + k^{-1}S(\omega_B)$. In general, one has

$$S(\omega_k) = S(\omega_F) + O(k^{-1}),$$

but the $O(k^{-1})$ term is more involved. In Lemma 2.2 below, we will prove why the expansion is this way.

The expansion $S(\omega_k) = S(\omega_F) + O(k^{-1})$ means that to leading order, the cscK equation is the cscK equation fibrewise. A natural assumption is therefore to start with a fibrewise cscK metric ω_Y (however, this will not be the case in the main result discussed today!). Note that there may be many such: if $Y = \mathbb{P}(E)$, any hermitian metric on E induces a relative Kähler metric on Y which is fibrewise Fubini-Study.

The non-uniqueness of the fibrewise cscK metric comes from two sources:

- Pullback of functions from B;
- Fibrewise holomorphic vector fields (cscK metrics are only unique up to automorphisms).

The former is harmless and should be thought of as analogous to Kähler potentials only being unique up to a constant in the absolute setting, but as we will see later, the latter causes many complications.

We want to construct extremal metrics on the total space of fibrations using perturbative techniques. Recall that the general strategy in such problems have the following key steps:

- Create approximate solutions to the extremal equation;
- Control the inverse of the linearised operator for the extremal equation;
- Apply a quantitative inverse/implicit function theorem, or contraction mapping theorem.

For the fibration setting, we will see that the control we get for the inverse of the linearised operator is not good enough that using just a fibrewise cscK metric together with some fixed pulled back metric from the base will allow us to use the implicit function theorem. We need a better approximate solution.

2. The case of fibrewise cscK metrics

We now want to discuss the method of proof in the case discussed above in more detail. Recall that the setup is that we have a holomorphic submersion

$$(Y, H) \to (B, D)$$

and the polarisation H is *fibrewise cscK*, i.e. there exists an $\omega_Y \in c_1(H)$ whose restriction to any fibre is cscK. The material in this section was developed by Fine in [**22**] and further refined by Dervan and the author in [**17, 19**] building also on the other works in the adiabatic setting mentioned above.

2.1. The expansion of the scalar curvature. We now prove exactly what the expansion of the scalar curvature is, to leading two orders. We will need to introduce some notation. Functions on Y split as

$$C^\infty(Y) = C^\infty_0(Y) \oplus \pi^* C^\infty(B),$$

where $C^\infty_0(Y)$ consists of fibrewise average 0 functions. We will sometimes use the notation ψ_0 to mean the $C^\infty_0(Y)$ component of a functions ψ on Y. We also have a splitting of the tangent bundle of Y as

$$TY = \mathcal{V} \oplus \mathcal{H}$$

where $\mathcal{V} = \ker \pi_*$ is the vertical tangent bundle and $\mathcal{H} \cong \pi^* TB$ is the horizontal tangent bundle. We have similar splittings of any tensor bundle.

We will let

- ρ be the curvature of $\Lambda^m \mathcal{V}$ induced by ω_Y^m;
- $\mathrm{Ric}(\omega_F)$ be the Ricci curvature of the metric induced on the fibres, which is simply the vertical component $\rho_{\mathcal{V}}$ of ρ;
- $\rho_{\mathcal{H}}$ be the horizontal component of ρ;
- $\Delta_{\mathcal{V}} = \Lambda_{\mathcal{V}} i\partial\bar\partial$ be the vertical Laplacian, where $\Lambda_{\mathcal{V}}$ is the contraction in the vertical direction;
- $F_{\mathcal{H}}$ be the curvature of the Ehresmann connection of the fibration $Y \to B$ given by the splitting $TY = \mathcal{V} \oplus \mathcal{H}$ of the tangent bundle of Y.

The term $F_{\mathcal{H}}$ is a two form with values in fibrewise hamiltonian vector fields. By using the comoment map μ^* taking the vector field to the corresponding hamiltonian of average 0 on the each fibre, we can view $F_{\mathcal{H}}$ as a two form on B with values in $C^\infty_0(Y)$. We can then contract in the base direction to get an element

$$\Lambda_{\omega_B}(\mu^* F_{\mathcal{H}}) \in C^\infty_0(Y).$$

One then has that

LEMMA 2.2.

$$S(\omega_k) = S(\omega_F) + k^{-1}(\psi_0 + \psi_B) + O(k^{-2}),$$

where

$$\psi_0 = (\Lambda_{\omega_B} \rho_{\mathcal{H}})_0 + \Delta_{\mathcal{V}}(\Lambda_{\omega_B}(\mu^* F_{\mathcal{H}})) \in C^\infty_0(Y)$$

and

$$\psi_B = \pi^* \left(S(\omega_B) - \Lambda_{\omega_B}(\alpha)\right) \in \pi^* C^\infty(B),$$

for some semipositive Weil–Petersson type 2-form α.

The geometric interpretation of the form α is that it is the pullback of the Weil–Petersson Kähler form on the moduli space $\mathcal{M}$ of cscK manifolds, via the map $B \to \mathcal{M}$ sending b to the cscK manifold $\big(Y_b, (\omega_Y)\big|_b\big)$. Note that the map to the moduli space is not necessarily an embedding, so α is only semipositive, not necessarily positive. For example, for isotrivial fibrations such as $\mathbb{P}(E)$, α is actually 0.

Proof of Lemma 2.2. It suffices to prove that we have the expansion

$$\mathrm{Ric}(\omega_k) = \mathrm{Ric}(\omega_F) + \rho_{\mathcal{H}} + \mathrm{Ric}(\omega_B) + k^{-1} i\partial\bar{\partial}(\Lambda_{\omega_B}\omega_Y) + O(k^{-2}).$$

Note that the vertical component of ρ is $\mathrm{Ric}(\omega_F)$. This expansion in turn implies that

$$\begin{aligned} S(\omega_k) &= S(\omega_F) + k^{-1}\big(\Lambda_{\omega_B}\rho_{\mathcal{H}} + S(\omega_B) + \Delta_{\mathcal{V}}(\Lambda_{\omega_B}(\omega_Y)_{\mathcal{H}}))\big) + O(k^{-2}) \\ &= S(\omega_F) + k^{-1}\big(\Lambda_{\omega_B}\rho_{\mathcal{H}} + S(\omega_B) + \Delta_{\mathcal{V}}(\Lambda_{\omega_B}(\mu^* F_{\mathcal{H}}))\big) + O(k^{-2}), \end{aligned}$$

by the expansion

$$\Lambda_{\omega_k} = \Lambda_{\mathcal{V}} + k^{-1}\Lambda_{\mathcal{H}} + O(k^{-2})$$

and since $\Lambda_{\omega_B}(\beta)$ is a pulled back function from the base, for any pulled back form β, and is hence in the kernel of $\Delta_{\mathcal{V}}$. The last line then follows because $\mu^* F_{\mathcal{H}}$ and $(\omega_Y)_{\mathcal{H}}$ differ by a form pulled back from B. The required expansion then also by nothing that the horizontal component of $\Lambda_{\omega_B}\rho_{\mathcal{H}}$ is the negative of the contraction of the Weil–Petersson form α.

The expansion

$$\mathrm{Ric}(\omega_k) = \mathrm{Ric}(\omega_F) + \rho_{\mathcal{H}} + \mathrm{Ric}(\omega_B) + k^{-1} i\partial\bar{\partial}(\Lambda_{\omega_B}\omega_Y) + O(k^{-2}).$$

for the Ricci curvature follows by using that

$$TY^{n+m} \cong \mathcal{V}^m \otimes \mathcal{H}^n,$$

coming from the splitting of the tangent bundle into vertical and horizontal components. Note that $\mathcal{H}^n \cong \pi^* TB^n$.

The $\mathcal{V}^m$ part in this decomposition comes from taking the vertical component of ω_k, which is the induced fibrewise metric that remains unchanged with k. This gives the term ρ, whose vertical component is $\mathrm{Ric}(\omega_F)$ and whose horizontal component is $\rho_{\mathcal{H}}$.

For the $\mathcal{H}^n$ part, note that we have another metric we can use on $\mathcal{H}^n \cong \pi^* TB^n$, namely the one induced by $\pi^*\omega_B$. We then have that the horizontal part is the curvature induced by $\pi^*\omega_B$, which is $\pi^* \mathrm{Ric}(\omega_B)$, plus $i\partial\bar{\partial}$ of the ratio of the volumes, computed in $\mathcal{H}^n$. This term is

$$\begin{aligned} i\partial\bar{\partial}\left(\log\frac{(\omega_k)^n_{\mathcal{H}}}{\omega_B^n}\right) =& i\partial\bar{\partial}\left(\log\frac{k^n\omega_B^n + k^{n-1}n(\omega_Y)_{\mathcal{H}}\wedge\omega_B^{n-1}+\dots}{\omega_B^n}\right) \\ =& i\partial\bar{\partial}\left(\log(1+k^{-1}\Lambda_{\omega_B}(\omega_Y)_{\mathcal{H}}+\dots)\right) \\ =& k^{-1}i\partial\bar{\partial}\Lambda_{\omega_B}(\omega_Y)_{\mathcal{H}} + O(k^{-2}). \end{aligned}$$

This completes the proof of the expansion of the Ricci curvature and hence of the scalar curvature, too. □

2.2. The expansion of the linearised operator. At this stage we see that we have a sequence of metrics ω_k on Y in Ω_k, which by the fibrewise cscK condition satisfies

$$S(\omega_k) = c + O(k^{-1}),$$

for a constant c. As remarked above, fibrewise cscK metrics are not necessarily unique, and at this stage we have used any fixed metric ω_B on the base. In the product case, it would only be when we pick a cscK metric on the base that we would solve the cscK equation on the product. In particular, if we in the product case pick a metric ω_B that is far away from cscK, we do not stand a chance in using perturbative techniques to produce a cscK metric on the product fibration. In general, therefore, we expect that there is at least a choice of ω_B that will have to come into play. Also, this suggests that one should expect that it is not enough to perturb from just the initial sequence of metrics ω_k.

We will now investigate how good of an approximate solution we will actually need in order to perturb to a genuine extremal metric on the fibration. This means establishing the mapping properties of the linearised operator. We will later see that these mapping properties will be important also to be able to create a good enough approximate solution, too.

The key lies in an asymptotic expansion of the linearised operator.

PROPOSITION 2.3 ([**19**, Proposition 4.11]). *The linearisation of the scalar curvature operator at ω_k admits an expansion*

$$dS(\omega_k)_0 = -L_F + k^{-1}D_1 + k^{-2}D_2 + O(k^{-3}),$$

where

- *L_F is the fibrewise Lichnerowicz operator of ω_Y;*
- *D_1 vanishes on base functions;*
- *the base component of the operator D_2, acting on base functions, is the linearisation of the twisted scalar curvature operator.*

The interpretation of this is that we can remove any decaying term orthogonal to fibrewise holomorphy potentials, and we can remove any base term that decays to order greater than k^{-2}. In fact, one can even do better: if one changes ω_B by $\omega_B + k^{-1}i\partial\bar{\partial}\phi$ for a function pulled back from ϕ, this changes the scalar curvature by the linearisation of the twisted scalar curvature at order k^{-1} and hence gives this contribution to the order k^{-2}-term of the scalar curvature of ω_k. Thus we can actually deal with any horizontal term decaying at order strictly greater than k^{-1}.

That the leading order term in the expansion is $-L_F$ means that we can remove any vertical term orthogonal to fibrewise holomorphy potentials using the linearisation. If the fibres have holomorphic vector fields, however, this is not the full space $C_0^\infty(Y)$. We will let $C_E^\infty(Y)$ denote the space of fibrewise average 0 holomorphy potentials with respect to ω_Y. Then the image of L_F is precisely the orthogonal complement to $C_E^\infty(Y)$.

To ensure that we can hit anything apart from global holomorphy potentials, we need to understand when we hit the whole of $C_E^\infty(Y)$. This is the content of the next result.

PROPOSITION 2.4 ([**19**, Theorem 4.9]). *The operator $q \circ D_1$, acting on $C_E^\infty(Y)$, can be identified with a self-adjoint elliptic operator on a vector bundle $E \to B$ with kernel precisely the fibrewise holomorphy potentials that are globally holomorphy potentials with respect to ω_k for any/all k sufficiently large.*

In particular, this means that we can deal with any vertical term orthogonal to global holomorphy potentials at order k^{-1}.

The above can be used to give a bound to the (right) inverse of the linearised operator of the metrics ω_k and perturbations thereof. For simplicity, we will assume that both B and Y have trivial reduced automorphism group, which in particular means that we are solving the cscK equation on the total space. Adjustments can be made in the case when this does not hold, and one is then solving the extremal equation instead, as Y has global holomorphic vector fields.

PROPOSITION 2.5 ([**22**, Proposition 6.5],[**19**, Proposition 5.6]). *The linearisation of the scalar curvature operator at ω_k (or a suitable perturbation of ω_k) has an inverse Q_k such that there for any j, α there exists a $C > 0$ such that*

$$\|Q_k\| \leq Ck^3$$

in operator norm, as an operator $C^{j,\alpha} \to C^{j+4,\alpha}$.

From the above proposition, we see that we need a better approximate solution in order to apply our perturbative techniques. We wish to use the following Quantitative Inverse Function theorem.

THEOREM 2.6. *Let $F : V \to W$ be a differentiable map of Banach spaces, whose derivative at 0 is an isomorphism with inverse η. Let*

- *r' be the radius of the ball in V centered at 0 in which $F - dF$ is Lipschitz of constant $\frac{1}{2\|\eta\|}$;*
- *$r = \frac{r'}{2\|\eta\|}$.*

Then for all $w \in W$ such that $\|F(0) - w\| \leq r$, there exists a $v \in V$ with $\|v\| \leq r'$ such that $F(v) = w$.

A quick calculation then shows that with the bound established in Proposition 2.5 we need an approximate solution that is extremal to up to and including order at least k^{-6} to apply the above theorem to produce a solution to the cscK/extremal equation.

2.3. The case of no automorphisms. We will now produce a solution to the extremal equation. From the above, we see that the initial sequence ω_k is not good enough to guarantee that we can perturb to an actual solution. We now explain how we improve the approximate solution so that it becomes approximately cscK or extremal to higher order. That is, we need to find new metrics $\omega_{k,l} \in [\omega_k]$ that are extremal to order k^{-l}. When ω_Y is fibrewise cscK, this gives us the case $l = 1$. As remarked above, we need to get to $l = 6$, but the process works to produce approximate solutions to arbitrary high order.

To improve this approximate solution, we go back to the analysis of the $O(k^{-1})$ term. Recall that we have an expansion

$$\begin{aligned} S(\omega_k) =& S(\omega_F) + k^{-1}(\psi_0 + \psi_B) + O(k^{-2}), \\ \psi_0 =& (\Lambda_{\omega_B}\rho_{\mathcal{H}})_0 + \Delta_{\mathcal{V}}(\Lambda_{\omega_B}(\mu^* F_{\mathcal{H}})), \\ \psi_B =& S(\omega_B) - \Lambda_{\omega_B}(\alpha). \end{aligned}$$

Looking at the latter term we see that, to improve the solution to $O(k^{-2})$, one therefore needs ω_B to solve a *twisted* cscK equation

$$S(\omega_B) - \Lambda_{\omega_B}(\alpha) = c_B$$

or more generally a twisted extremal equation

$$S(\omega_B) - \Lambda_{\omega_B}(\alpha) \in \overline{\mathfrak{h}}_B,$$

where $\overline{\mathfrak{h}}_B$ denotes the holomorphy potentials on B with respect to ω_B. Recall that we can only remove horizontal terms at order strictly greater than k^{-1}, so to deal with the $O(k^{-1})$ we need to make a good choice of metric to pull back from B. Assuming one can find such an ω_B, we have now determined ω_B: but we still have some freedom as we can perturb!

We therefore need to understand the contribution to the above if we perturb to $\omega_{k,2} = \omega_k + k^{-1} i\partial\bar{\partial}\phi$, for some ϕ. This boils down to understanding the linearisation P_k of the scalar curvature operator, at ω_k, which was the content of Proposition 2.3 of the previous section. Recall that the linearisation has an expansion

$$P_k = -L_F + O(k^{-1}),$$

where L_F is a fibrewise Lichnerowicz operator, if ω_Y is fibrewise cscK.

Now, if the fibres have no automorphisms, L_F is surjective on $C^\infty_0(Y)$. We can therefore remove the ψ_0 term, and continue to improve the solution to arbitrary order, also using that any horizontal term at order strictly greater than k^{-1} can be removed by perturbing, using the expansion of the linearisation. Eventually, the approximate solutions are sufficiently good so that the Inverse Function Theorem 2.6 can be applied to find a solution to the extremal equation for $k \gg 0$.

2.4. What if the fibres have automorphisms? If the fibres have automorphisms, we cannot remove the ψ_0 term completely using the linearisation. However, in this case, there is also some choice of ω_Y, since the metric is not uniquely determined on each fibre. Indeed, a theorem of Berman–Berndtsson shows that extremal metrics are only unique up to the action of the automorphism group ([**6**]). So while it is not automatic that we can remove the ψ_0 term, we also have many fibrewise cscK metrics to consider initially.

In [**19**], Dervan and the author found an equation for a good choice of fibrewise cscK metric. The equation is

$$q\left(\Lambda_{\omega_B}\rho_{\mathcal{H}} + \Delta_{\mathcal{V}}(\Lambda_{\omega_B}(\mu^* F_{\mathcal{H}}))\right) = 0,$$

where q is the fibrewise L^2-orthogonal projection to average 0 holomorphy potentials. This says that precisely the term that cannot be removed at the crucial stage using the linearisation vanishes. This is an equation on fibrewise cscK metrics (actually, an important fact in the theory is that one can view these as smooth sections of a finite dimensional vector bundle over X). In the case when $Y = \mathbb{P}(E)$, this reduces to the Hermite–Einstein equation for a hermitian metric on E.

Under the assumption that such a metric ω_Y exists, the construction still goes through, following the same type of steps as in the case of trivial automorphism groups of the fibres. This requires one to use the subleading order terms in the asymptotic expansion of the vertical part of the linearised operator, see Proposition 2.3 and Proposition 2.4. From this, one sees that at order k^{-1}, one can hit anything orthogonal to global holomorphy potentials. This means that any vertical error after the crucial term at order k^{-1} can be removed, which is the essential step in constructing a good approximate solution.

In the proper statement of the main result, we will make some assumptions on the automorphism groups. This is essentially to avoid running into these issues, which significantly complicate the construction.

3. The new construction

3.1. Statement of the result. The new construction produces extremal metrics on a special type of fibration. We have

- $(Y, H) = (\mathcal{X}, \mathcal{L})$, the total space of a test configuration for some (X, L) over $\mathbb{P}^1$, satisfying certain assumptions;
- $(B, D) = (\mathbb{P}^1, \mathcal{O}(1))$.

If $(\mathcal{X}, \mathcal{L})$ is a product test configuration and (X, L) admits a cscK metric, it is already known that one can produce extremal metrics in the adiabatic classes $\mathcal{L} + \mathcal{O}(k)$. This follows e.g. by the construction of Dervan–S. detailed above.

In the construction under discussion now, however, we will have that *almost no fibre* actually admits a cscK metric. This is in stark contrast to the previous constructions. For us, (X, L) will be a strictly K-semistable manifold, and the degeneration above is to a cscK central fibre $(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0})$.

More precisely, we consider a Kähler manifold (X, L) which is *analytically K-semistable*. This means that there is a $\mathrm{Aut}_0(X)$-equivariant degeneration $(\mathcal{X}, \mathcal{L})$ of (X, L) to a cscK Kähler manifold $(\mathcal{X}_0, \mathcal{L}|_{\mathcal{L}_0})$, see Section 3.2 below for more details on precisely the test configurations we consider. By [**21, 37, 15**], the condition implies K-semistability of (X, L). It can also be seen as asking that (X, L) is a small equivariant deformation of a cscK manifold.

The main result with Spotti is the following

THEOREM 2.7 ([**32**]). *Suppose (X, L) is analytically strictly semistable and has discrete automorphism group. Suppose $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{P}^1$ is a test configuration degenerating (X, L) to a cscK central fibre $(\mathcal{X}_0, \mathcal{L}|_{\mathcal{L}_0})$. Assume further that the reduced automorphism group of the central fibre is $\mathbb{C}^*$. Then for all large k, there exists an extremal metric in $c_1(\mathcal{L} + \pi^*\mathcal{O}(k))$.*

The main importance of the result is two-fold: first of all, it is the first construction where one is able to remove the condition that all the fibres admit cscK/extremal metrics. The construction therefore provides an important technique that allows dealing with the semistable case, where no canonical choice of fixed fibrewise metric will exist. Secondly, the construction actually provides a myriad of new extremal metrics. We will discuss this in Section 4.

REMARK 2.8. The assumption regarding the automorphism group can be thought of as analogous to the assumption that the automorphism group is discrete in similar perturbation problems. The central fibre has to have a $\mathbb{C}^*$ action, and so the assumption says that the automorphism group is minimal. In particular, we see no obstructions, unlike the cases related to the OSC equation. The freedom we have in the choice of cscK metric on the central fibre, precisely equals the freedom we have on $\mathcal{X}$, which also has a $\mathbb{C}^*$ as its automorphism group.

REMARK 2.9. We actually relax the condition on the automorphism group somewhat. We will not go into details of this, but the condition is really that the difference between the reduced automorphism group of the central fibre and the original semistable manifold is given by the additional $\mathbb{C}^*$ action on the central fibre.

3.2. Kuranishi theory. We give slightly more details on how exactly we assume the test configuration arises. This uses the Kuranishi theory of the cscK central fibre, following the exposition of [**39**].

Suppose (X_0, L_0) is a polarised manifold with cscK metric $\omega \in c_1(L)$, and whose underlying smooth manifold is M. Let T be a maximal compact torus in the reduced automorphism group of X_0. Then there exists a complex space V with a T-action and a holomorphic embedding $V \to \mathcal{J}(M,\omega)$, equivariant with respect to the T-action, which induces a versal deformation space for X_0. One can moreover ensure that the scalar curvature of the complex structures in the family (with respect to the form ω) takes values in the holomorphy potentials of (X_0, ω) ([**39, 7**]).

It is certain test configurations produced from this family that we will consider in our construction. The key fact we are using is that the versality of the construction implies that if we have a test configuration taking some (X, L) to a cscK (X_0, L_0), then there exists a (potentially different) test configuration $(\mathcal{X}, \mathcal{L})$ in the Kuranishi family, taking (X, L) to (X_0, L_0). It is on this family that our construction takes place. That is, the test configurations $\mathcal{X}$ we consider are produced from the Kuranishi family near the polystable (X_0, L_0).

The upshot is that we can assume that we have a fixed smooth manifold M with a symplectic form ω and family J_t with t some disk Δ about 0 in $\mathbb{C}$, such that for (M, J_0, ω) is cscK, and $(M, J_t) \cong X$ for $t \neq 0$. Moreover, $J_t = J_0 + O(|t|)$, and the $O(|t|)$ term is non-vanishing. The relationship between $\mathcal{X} \to \Delta$ and J_t is that as a smooth manifold, $\mathcal{X} = \Delta \times M$ and J_t is the complex structure of the fibre of $\mathcal{X}$ over t in Δ.

3.3. The proof. Recall the old approach for a cscK fibration $(Y, H) \to (B, D)$: one starts with a metric $\omega_Y \in c_1(H)$ and metric $\omega_B \in c_1(D)$ and considers the one parameter family of metrics $\omega_k \in c_1(H + kD)$ given by

$$\omega_k = \omega_Y + k\omega_B.$$

If ω_Y is fibrewise cscK, then we obtain

$$S(\omega_k) = c + O(k^{-1}).$$

The problem in our case $(Y, H) = (\mathcal{X}, \mathcal{L})$, is that there are no fibrewise cscK metrics in $c_1(\mathcal{L})$: all but one fibre does *not* admit a cscK metric! Thus, if we work with any fixed metric $\omega_{\mathcal{X}} \in c_1(\mathcal{L})$, we can never hope to obtain good approximate solutions.

Upshot: we need to work with a *sequence* of relative Kähler metrics $\widetilde{\omega}_k \in c_1(\mathcal{L})$.

To explain why there is some hope to get a good sequence of relative Kähler metrics, we recall the following conjecture of Donaldson.

CONJECTURE 2.10 ([**21**]). *Let $(V, \mathcal{D})$ be a polarised projective manifold. Then*

$$\inf_{\omega \in c_1(\mathcal{D})} \|S(\omega) - \hat{S}\| = \sup_{\mathcal{V} \text{ t.c.}} \frac{-DF(\mathcal{V})}{\|\mathcal{V}\|}.$$

In particular, if $(V, \mathcal{D})$ is strictly semistable, we should be able to find metrics that are *arbitrarily close to cscK*, even though no actual cscK metric exists.

Under our assumptions, the fact that we can get a family of metrics $\alpha_\varepsilon \in c_1(L)$ satisfying

$$\|S(\alpha_\varepsilon) - \hat{S}\| \leq \varepsilon$$

on the non-cscK general fibre X is automatic. This comes from the Kuranishi theory detailed above: since $J_\varepsilon = J_0 + O(\varepsilon)$,

$$S(\omega, J_\varepsilon) = S(\omega, J_0) + O(\varepsilon).$$

If we then use a diffeomorphism to identify the complex structure of the non-zero fibres with a fixed fibre and pullback the symplectic form, we get the symplectic forms α_ε, and these have the same scalar curvature as (ω, J_ε), as these are isomorphic Kähler manifolds.

Next, we want to take our local family over some disk Δ and extend it to a relative Kähler metric on the corresponding compactified test configuration over $\mathbb{P}^1$, which we still call $\mathcal{X}$. As the metrics are *not* necessarily identical on the S^1-fibres, this is not automatic. We then use a rather crude interpolation so that

$$\omega_\varepsilon|_{\mathcal{X}_t} = \begin{cases} \alpha_t & \text{if } t \leq \frac{\varepsilon}{2} \\ \alpha_\varepsilon & \text{if } t \geq \varepsilon \end{cases}$$

In the trivialisation identifying the test configuration with $\mathbb{C}^* \times X$ over $\mathbb{C}^*$, we therefore have a constant metric on every fibre, outside the disk of radius ε. We can then trivially extend over $\infty \in \mathbb{P}^1$, to obtain a relative Kähler metric on the full family. This metric satisfies that

$$\|S(\omega_\varepsilon|_{\mathcal{X}_t}) - \hat{S}\| \leq C\varepsilon$$

for *every* fibre $\mathcal{X}_t$.

We now want to relate the parameter ε in our family of relative Kähler metrics on $\mathcal{X}$, to the parameter k of the polarisation. If we write

$$\Omega_\varepsilon = \omega_\varepsilon + \lambda\varepsilon^{-\delta}\pi^*\omega_{\mathbb{P}^1},$$

for some Kähler form $\omega_{\mathbb{P}^1}$ on $\mathbb{P}^1$, so $k = \lambda\varepsilon^{-\delta}$, then this is Kähler for all sufficiently small ε if either

- $\delta = 1$ and λ is sufficiently large;
- $\delta > 1$.

For us, it suffices to let $\delta > 1$, and we can even take $\delta = 2$. The fact that we can get away with not have to pick a specific value of δ is related to the our assumptions on the automorphism group.

The point of picking ω_ε the way we did is that now

$$S(\Omega_\varepsilon) = S(\omega, J_0) + O(\varepsilon).$$

I.e., we are seeing the scalar curvature of the *central fibre* as the leading order term in the expansion of the scalar curvature. We have lower order vertical terms we have to deal with, but these cause no problem. The expansion of the Lichnerowicz operator L_ε of Ω_ε is

$$L_\varepsilon = L_0 + O(\varepsilon),$$

where L_0 is the Lichnerowicz operator of the central fibre. This means that we can remove any error orthogonally to the generator of the $\mathbb{C}^*$ action on central fibre – but this comes from a global vector field on $\mathcal{X}$! So to solve the extremal equation on $\mathcal{X}$, this poses no obstruction. Note that it does pose an obstruction to the cscK equation on $\mathcal{X}$, but this obstruction is precisely the classical Futaki invariant of the $\mathbb{C}^*$ action on the total space of the test configuration.

REMARK 2.11. When the discrepancy between the automorphism group of the central fibre and the general fibre is larger, more care is needed in the construction. This is analogous to the setting of the Optimal Symplectic Connection equation, where we need to use the extra freedom of the choice of cscK metric on the central fibre, to make up for the terms we cannot remove using the linearisation. It is then likely that one needs a more careful extension process, and to work with the critical exponent δ for when the metrics are Kähler. This likely changes the equation one needs to solve on the base as well.

The tools to extend to this setting are on the way. A general theory extending the optimal symplectic connection equation to fibrations with semistable fibres is currently being developed by Annamaria Ortu.

4. Examples

The construction allows us to produce many new examples of extremal metrics. The key to producing examples is to produce explicit examples of K-semistable manifolds that degenerate to a *smooth* cscK central fibre. This means a degeneration of (X, L) to a cscK central fibre exists in the Kuranishi family, by versality, and this produces the test configuration $\mathcal{X}$ we actually work on. Once one has such a degeneration, only the $\mathbb{C}^*$ discrepancy condition for the reduced automorphism group has to be verified.

In general such examples can be difficult to produce, but in the setting of Fano remarkable progress has been made through work on the valuative criterion for stability, which has allowed one to actually check K-stability and thus verify whether or not there exists a Kähler–Einstein metric. This has been particularly fruitful on Fano threefolds, through the work of a wide group of researchers. These developments are described in the book [**2**], and we refer to it and its references for details on what we will mention.

There are 105 deformation families of Fano threefolds, and the collection of these is referred to as the Mori–Mukai list of Fano threefolds. In most of these families, either all members are K-polystable or are strictly K-unstable. However, there exists some families with both K-polystable and K-unstable members, including strictly K-semistable ones, which is the most important for our purposes.

In order to produce a test configuration to which the construction applies we need to verify two things: that we have strictly K-semistable manifold degenerating to a *smooth* K-polystable central fibre, and that the discrepancy between the reduced automorphism groups of the general fibre and central fibre is correct. The latter part relies on the work of [**11**].

As an illustration of the method, we present one example here. All the explicit families we consider are described in [**2**].

LEMMA 2.12. *Let X be the Fano threefold in the family 4.13 of the Mori–Mukai list given as the blowup of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, with homogenous coordinates $([x, y], [u, v], [p, q])$, in the curve given by the two equations*

$$xv - yu = x^3p + y^3q + xy^2p = 0.$$

Then there exists a test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, -K_X)$ for which the construction of Theorem 2.7 applies.

PROOF. By [**2**, Corollary 5.22.3], X admits a test configuration degenerating X to the Fano threefold X_0 given as the blowup of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in the curve

$$xv - yu = x^3p + y^3q = 0$$

simply by scaling the latter equation. That is, the test configuration is given by the family obtained by blowing up in

$$xv - yu = x^3p + y^3q + sxy^2p = 0$$

for $s \in \mathbb{C}$. The central fibre is K-polystable by [**2**, Theorem 5.22.7], and by [**11**] has automorphism group $\mathbb{C}^*$, while X has trivial automorphism group. It follows that the construction applies to a test configuration for $(X, -K_X)$. □

Similar techniques show that the following families produce at least one test configuration to which the construction applies. In some of these cases, we produce infinitely many examples, and in others, only one.

THEOREM 2.13 ([**32**, Theorem 6.8]). *The following families from the Mori–Mukai list of Fano threefolds produce at least one test configuration for a strictly K-semistable manifold to which the construction applies:*

- *1.10*
- *2.20*
- *2.21*
- *2.22*
- *2.24*
- *3.5*
- *3.8*
- *3.10*
- *3.12*
- *4.13*

Together, they give infinitely many projective manifolds that admit extremal Kähler metrics in some classes.

Remark 2.14. The above result says that in each of the cases mentioned, we have found at least *one* member, to which we can apply our construction. We have not checked every possible member of the family in order to get a complete classification. Also, not all the families will give test configurations to which we can apply the construction, even if there are strictly K-semistable members of these families. This can come both from there being no smooth K-polystable member in the family, or that the discrepancy condition on the reduced automorphism groups does not hold.

LECTURE 3

Extremal metrics on blowups

In the final lecture, we will discuss recent work with Ruadhaí Dervan ([**18**]), which concerns the construction of extremal metrics on the blowup of a manifold in a point.

1. The question we want to answer

Let X be a compact Kähler manifold, and let $p \in X$ be a point. We can then define the blowup

$$\pi : \mathrm{Bl}_p X \to X$$

of X in p. This is a manifold satisfying:

- It is isomorphic to $X \setminus \{p\}$ outside the preimage of p;
- The preimage of p via the blowdown map π is a copy E of $\mathbb{P}^{n-1}$, called the exceptional divisor.

We will now define it.

The local model is the blowup $\mathrm{Bl}_0 \mathbb{C}^n$ of $\mathbb{C}^n$ in the origin. This can be seen as the total space of $\mathcal{O}(-1)$ over $\mathbb{P}^{n-1}$. In other words,

$$\begin{aligned} \mathrm{Bl}_0 \mathbb{C}^n =& \{([z], v) : v = \lambda z \text{ for some } \lambda \in \mathbb{C}\} \\ \subset& \mathbb{P}^{n-1} \times \mathbb{C}^n \end{aligned}$$

The map to the second factor is an biholomorphism away from the origin. At the origin, the fibre is $\mathbb{P}^{n-1}$. Thus we have replaced the origin with a divisor, the exceptional divisor, describing all the complex directions we can enter the origin in. The name comes from thinking of this as "zooming in" on the origin – we have "blown up" near the point.

For a general complex manifold X, the blowup in p is the complex manifold obtained by replacing a disk about p with the preimage in $\mathrm{Bl}_0 \mathbb{C}^n$ of a disk about the origin in $\mathbb{C}^n$. Since the local model is a biholomorphism away from the origin, we can view this as a gluing of $X \setminus \{p\}$ and the preimage of the disk. The resulting manifold is therefore a complex manifold. As with the local model, we have a map

$$\pi : \mathrm{Bl}_p X \to X$$

and the preimage of p via π is a copy E of $\mathbb{P}^{n-1}$.

Next, note that

$$H^2(\mathrm{Bl}_p X) \cong H^2(X) \oplus \langle [E] \rangle.$$

Moreover, if Ω is a Kähler class, then $\pi^*(\Omega)$ is on the boundary of the Kähler cone of $\mathrm{Bl}_p X$ – all subvarieties have non-negative volume, but the exceptional divisor has volume 0. So the class cannot be Kähler. However, if we allow the exceptional divisor to get some positive volume, we move into the Kähler cone of $\mathrm{Bl}_p X$. That is, for all $\varepsilon > 0$ sufficiently small, the class

$$\Omega_\varepsilon = \pi^*\Omega - \varepsilon[E]$$

is a Kähler class on $\mathrm{Bl}_p X$.

The question we want to answer is the following: under what conditions on (X, Ω) and p does $(\mathrm{Bl}_p X, \Omega_\varepsilon)$ admit a cscK or extremal metric for all $\varepsilon > 0$ sufficiently small? Note, we are not trying to understand what happens for every value of ε such that Ω_ε is Kähler. We are only trying to understand what happens when ε is very small, i.e. when the volume of the exceptional divisor is very small.

2. Prior work

This question has a rich history of study, through works of Arezzo–Pacard ([**3, 4**]), Arezzo–Pacard–Singer ([**5**]) and Székelyhidi ([**40, 42**]) giving sufficient conditions for when the blowup does admit an extremal metric in these classes, and Stoppa [**35, 36**] and Stoppa–Székelyhidi [**37**] for necessary conditions. Seyyedali–Székelyhidi ([**33**]) have also considered the case of blowing up higher dimensional subvarieties (see also [**25**] for prior work on the special case of blowing up projective space in a line). We will now explain some of the history and known facts before the work [**18**] of Dervan and the author.

The initial case considered by Arezzo–Pacard in [**3**] is when X admits a cscK metric in Ω and has trivial reduced automorphism group. In this case, any point will do in the construction.

THEOREM 3.1 ([**3**]). *Suppose X admits a cscK metric in Ω. Let $p \in X$ be any point. Then there exists a $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\mathrm{Bl}_p X$ admits a cscK metric in the class $\Omega_\varepsilon = \pi^*\Omega - \varepsilon[E]$.*

On the other hand, in [**35, 36**], Stoppa showed that if one has a test configuration for some (X, L), this induces a test configuration for the blowup of X. He then analysed the Donaldson–Futaki invariant of these test configurations with respect to the polarisations making the exceptional divisor small. The leading order term in the expansion is the Donaldson–Futaki invariant of the original test configuration before the blowup. In particular, if (X, L) is strictly unstable, so will the blowup be, in the polarisations we consider. Moreover, he used the expression for the subleading order term to show that for any strictly K-semistable manifold one can find a point such that the blowup in that point is strictly unstable.

Thus in the situation of trivial reduced automorphism group, the only case not fully considered is when X is K-semistable, but does not admit a cscK metric.

The situation quickly gets more complicated in the presence of automorphisms, however. The simplest case in the presence of automorphisms is when blowing up a fixed point of a maximal torus in the reduced automorphism group of X. Recall that the reduced automorphism group of $\mathrm{Bl}_p X$ can be seen as the subgroup of the reduced automorphism group of X generated by the vector fields that vanish at the blown up point. So, in this case, all the relevant vector fields lift (we only need to work with torus-invariant functions), and, in fact, the extremal equation is unobstructed in this case.

THEOREM 3.2 ([**4, 5, 40**]). *Suppose X admits an extremal metric in Ω. Let $p \in X$ be a point fixed under the action of a maximal torus in the reduced automorphism group of X. Then there exists a $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\mathrm{Bl}_p X$ admits an extremal metric in the class $\Omega_\varepsilon = \pi^*\Omega - \varepsilon[E]$.*

The issue comes when we want to blow up a point not fixed by a maximal torus. This is when we see start to see obstructions to the equation, coming from vector fields that do not lift. The works of Arezzo–Pacard and Arezzo–Pacard–Singer used a gluing method where they consider $\mathrm{Bl}_p X$ as two manifolds with boundary glued along the boundary. They then use a "Cauchy matching" technique, to give sufficient conditions for the blowup to admit an extremal metric. The conditions appearing in their work come from the conditions needed to perform this matching to subleading order. In fact, they deal with multiple points simultaneously, and when blowing up a single point, their method only works when the point is fixed under the action of a maximal torus.

In [**40, 42**], Székelyhidi used a different approach, closer to the general strategy that we have discussed, which allowed him to relate the conditions on the point to K-stability. In the cscK case, when X has dimension at least 3, he showed that the blowup admits a cscK metric if and only if the manifold is K-polystable, and moreover gave a finite dimensional GIT condition that captures precisely what is needed to check K-polystability in this case. As this method forms the core of the method used by Dervan and the author in their approach to the problem as well, we will discuss in greater detail below how this approach works.

Finally, we also mention that the Donaldson–Futaki invariant and relative K-stability of blowups has been investigated by Stoppa–Székelyhidi in [**37**]. The blowup of a strictly relatively unstable manifold is relatively unstable and, again, the case of a relatively semistable manifold that does not admit an extremal metric is proved to be relatively unstable (strictly) for some choice of point.

The conclusion is that after the works mentioned above, the cases left unresolved are in the cscK case in dimension, the non-cscK extremal case in arbitrary dimension and the general strictly semistable case. In [**18**], we complete the work on the first two cases, and also prove similar results for *analytically* semistable manifolds.

3. Székelyhidi's approach

We will now discuss Székelyhidi's approach. The case that will be most important for us is when blowing up an extremal manifold in a fixed point of the action of a maximal torus. We will mostly consider this case and then remark on why there are obstructions appearing when blowing up a point that is not fixed at the end of the section. The argument is an outline of the main steps only, and we refer to [**40**] as well as [**41**] for more details.

We will consider $\mathrm{Bl}_p X$ as the union of two non-compact manifolds:

- $X \setminus \{p\}$;
- the preimage $\pi^{-1}(D)$ of a large disk $D \subset \mathbb{C}^n$ via the blowdown map $\mathrm{Bl}_0 \mathbb{C}^n \to \mathbb{C}^n$.

We will equip $X \setminus \{p\}$ with an extremal metric ω from X invariant under the corresponding real maximal torus, and $\mathrm{Bl}_0 \mathbb{C}^n$ with a certain asymptotically flat, scalar-flat metric η, the *Burns–Simanca metric* ([**34**]). Over an annular region (whose size depends on ε), we will then interpolate between the two metrics at the level of Kähler potentials to create an initial approximate solution to the equation.

This gluing is performed as follows. First of all, note that we can pick holomorphic normal coordinates $(z_1, \ldots, z_n)$ at p in which the torus action becomes a usual linear action. After rescaling the extremal metric ω on X, we may assume that the coordinates are defined on the ball of radius 2. We will let $(w_1, \ldots, w_n)$ be coordinates on $\mathrm{Bl}_0 \mathbb{C}^n$ away from the exceptional divisor, and glue the two coordinate systems via the coordinate change

$$w = \varepsilon^{-1} z.$$

Now, locally around p, the extremal metric ω can be written

$$\omega = i\partial\bar{\partial}\left(|z|^2 + \phi(|z|)\right),$$

for a function ϕ which is $O(|z|^4)$, since we have chosen holomorphic *normal* coordinates at p. Similarly, the Burns–Simanca metric η satisfies that

$$\varepsilon^2 \eta = i\partial\bar{\partial}\left(|z|^2 + \varepsilon^2 \psi(\varepsilon^{-1} z)\right)$$

under the coordinate change $w = \varepsilon^{-1} z$. This uses that the metric is asymptotically flat.

We can then interpolate between the two metrics on the level of potentials over an annular region

$$D_{2r_\varepsilon} \setminus D_{r_\varepsilon} = \{z : r_\varepsilon < |z| \leq 2r_\varepsilon\},$$

where we choose $r_\varepsilon = \varepsilon^{\frac{n-1}{n}}$. This is achieved as follows. Pick a smooth cut-off function $\chi(t)$ which vanishes on $t \leq 1$ and is equal to 1 when $t \geq 2$. Let $\chi_1(z) = \chi(\frac{z}{r_\varepsilon})$ and $\chi_2 = 1 - \chi_1$. Then we define ω_ε to be

- ω on $X \setminus D_{2r_\varepsilon}$;
- $\varepsilon^2 \eta$ on $\pi^{-1}(D_{r_\varepsilon})$;
- $i\partial\bar{\partial}\left(|z|^2 + \chi_1(z)\phi(z) + \varepsilon^2 \chi_2(z)\psi(\varepsilon^{-1} z)\right)$ on $D_{2r_\varepsilon} \setminus D_{r_\varepsilon}$.

One can show that ω_ε is Kähler when ε is sufficiently small. Moreover, we can obtain a uniform bound for the scalar curvature. The analysis all takes place in certain weighted norms $C^{k,\alpha}_\delta$ that measure the blowup or vanishing rate of functions near the exceptional divisor. To avoid too many technicalities, we will note define these spaces, but instead refer to [**40**] for the definitions.

Let $\hat{S}$ be the average scalar curvature of ω and let H_ε be the holomorphy potential of average 0 with respect to ω_ε of the lift of the extremal vector field on X to $\operatorname{Bl}_p X$. One then has the following uniform bound for scalar curvature of ω_ε.

LEMMA 3.3 ([**40**]). *For all $\delta < 0$, there exists a $C > 0$ such that for all sufficiently small positive ε,*

$$\|S(\omega_\varepsilon) - H_\varepsilon - \hat{S}\|_{C^{k,\alpha}_\delta} \leq C r_\varepsilon^{-\delta}$$

Thus ω_ε is an approximately extremal metric on $\operatorname{Bl}_p X$ and we wish to perturb this to a genuine extremal metric.

In order to perturb, we need to control the right inverse to the linearised operator. Since we are blowing up at a fixed point of the torus action, all the holomorphic vector fields on X lift and so the (co)-kernel of the Lichnerowicz operator L_ε of ω_ε can be identified with that of the Lichnerowicz operator L of ω on X.

PROPOSITION 3.4 ([**40**]). *Assume the dimension is at least 3. Then for $\delta \in (4-2n, 0)$, the operator $C^{k+4,\alpha}_\delta \times \mathfrak{h} \times \mathbb{R} \to C^{k,\alpha}_{\delta-4}$ given by*

$$(\phi, h, c) \mapsto L_\varepsilon(\phi) - h_\varepsilon - c$$

is surjective with right inverse Q_ε satisfying the uniform estimate

$$\|Q_\varepsilon\|_{C^{k,\alpha}_{\delta-4} \to C^{k+4,\alpha}_\delta} \leq C$$

for some $C > 0$.

When the dimension is 2, the bound for the right inverse blows up with ε. This causes some additional complications in the argument that we will not get further into here (one has to improve the approximate solution by matching closer with the Burns–Simanca metric). Thus for the remainder of the lecture, we will assume the dimension is at least 3, even though the case of dimension 2 also can be dealt with.

We now wish to show that we can perturb the approximate solution ω_ε to a genuine solution of the extremal equation. We want to use the contraction mapping theorem, and so we need to rephrase the equation as a fixed point problem. Let

$$\Phi : C^{k+4,\alpha}_\delta \times \mathfrak{h} \times \mathbb{R} \to C^{k+4,\alpha}_\delta \times \mathfrak{h} \times \mathbb{R}$$

be given by

$$(\phi, h, c) \mapsto Q_\varepsilon \left(S(\omega_\varepsilon + i\partial\bar{\partial}\phi) - h_\varepsilon - c - H_\varepsilon - \hat{S} \right).$$

We subtract off $H_\varepsilon + \hat{S}$ to make $(0,0,0)$ an approximate solution to the equation we wish to solve, through Lemma 3.3.

Székelyhidi then shows that there is a constant $c > 0$ such that Φ is a contraction on the set

$$U = \{(\phi, h, c) : \|(\phi, h, c)\|_{C^{k+4,\alpha}_\delta} \leq C\varepsilon^{2-\delta}\}.$$

Since $(r_\varepsilon)^{4-\delta} = \varepsilon^{(4-\delta)\frac{n-1}{n}}$ and $(4-\delta)\frac{n-1}{n} > 2 - \delta$ when $\delta < 0$ is sufficiently close to 0, we have that the Φ sends U to itself for a suitably chosen value of the parameter δ, for all sufficiently small ε. Hence the contraction mapping theorem guarantees the existence of an extremal metric on $\mathrm{Bl}_p X$ in $[\omega_\varepsilon]$ when $\varepsilon > 0$ is sufficiently small, which is what we wanted to show.

We end the section by briefly explaining what happens when the point is not a fixed point of the maximal torus. In this case, not all vector fields will lift. Székelyhidi defines a lift

$$l : \overline{\mathfrak{h}} \to C^\infty(\mathrm{Bl}_p X)$$

of the corresponding holomorphy potentials by using cut-off functions. These are no longer holomorphy potentials on the blowup. When establishing the properties of the linearisation analogous to Proposition 3.4 in this case, one can only establish a uniform bound orthogonally to $l(\overline{\mathfrak{h}})$. Since not all these functions are holomorphy potentials, this means that we are solving a more general equation than the extremal equation, and the construction is therefore obstructed. Székelyhidi then proceeds to understand this obstruction in terms of K-stability. Moreover, to get precise information on the relation to K-stability and a finite-dimensional GIT condition, a better approximate solution with higher matching with the Burns–Simanca metric is required. The need to have this higher order matching is why Székelyhidi's results are restricted to the cscK case and dimension at least 3.

4. The new approach – initial setup and statement of result

We now explain the core new idea in the work of Dervan and the author. The idea is to work with a fixed *symplectic* manifold in the gluing process, and let the complex structure vary instead. When blowing up a complex manifold, the complex structure will depend on the point chosen. On the other hand, in the symplectic category, we have no dependence on the point. As a smooth manifold, the blowup is $X \# \overline{\mathbb{CP}^n}$, and we will later see that we can make the symplectic form the same independently of the point, at least in an open neighbourhood on X. Of course, the *almost complex structure* will then change when we do this.

Our goal is to produce a diffeomorphism $f : X \to X$ sending a point p fixed by the action of a maximal torus to a nearby point q, in such a way that $f^*\omega = \omega$, i.e. such that f is a symplectomorphism. If $X = (M, J)$, where M is the underlying smooth manifold and J is the almost complex structure, we can then define $J_q = f^*J$. The blowup of (M, J_q) will then be isomorphic to the blowup of X in q, and as since f is a symplectomorphism, we even have that the Kähler manifolds (M, J_q, ω) and $(X, \omega) = (M, J, \omega)$ are isomorphic as Kähler manifolds. The advantage is that in the construction we can now take the point of view that we are blowing up a fixed symplectic manifold (M, ω) in one given point p. The thing that is changing is the almost complex structure, rather than the point.

In order to achieve this, we need to produce the symplectomorphism f. The key is to use Moser's trick: if β is a 1-form, then by flowing along the vector field dual to β via ω, we can produce a diffeomorphism $f : M \to M$ such that $f^*(\omega + d\beta) = \omega$. But if β is closed, $\omega + d\beta = \omega$, so then f will be a symplectomorphism. The goal is then to pick a good β so that p is sent to q.

To achieve this, we consider the local model of blowing up the origin in $\mathbb{C}^n$ with its Euclidean symplectic form. Suppose q is the point $(\lambda_1, \dots, \lambda_n)$. The flow generated by the real vector field ν corresponding to

$$\sum_{j=1}^{r} \lambda_j \frac{\partial}{\partial z^j}$$

sends the origin to $(\lambda_1, \dots, \lambda_n)$. This vector field is dual via the Euclidean symplectic form to $d(\sum_{j=1}^{r} \lambda_j |z^j|^2)$. Using cut-off functions, we can globalise this to an exact form β generating a Hamiltonian symplectomorphism $f : M \to M$. Because the Kähler metric is not exactly the Euclidean metric, f does not necessarily send p to the point q, but it will send p to a point q' such that $\mathrm{Bl}_{q'} X \cong \mathrm{Bl}_q X$, which suffices for our purposes. Thus from the local model we can produce a global symplectomorphism that does what we want.

We can also let the complex structure vary even before changing the point, and this allows us to tackle also certain strictly semistable manifolds. Recall from the previous lecture that we say that X is *analytically K-semistable* if it admits a degeneration, invariant with respect to the reduced automorphism group, to a cscK central fibre. Similarly, one says X is analytically relatively K-semistable if the same holds, but where the central fibre is extremal.

Thus we have a map

$$\Psi : B \to \mathcal{J}(M, \omega), \tag{4.1}$$

where B is an open ball in some vector space, which parametrises the isomorphism classes of complex structures near the cscK central fibre X_0 in the Kuranishi model, and all nearby points to a point p fixed by a maximal torus in the reduced automorphism group of the cscK central fibre.

Ultimately, we prove the following.

THEOREM 3.5 ([**18**]). *Suppose X is analytically relatively K-semistable, and let $p \in X$. Let $\Omega_\varepsilon = \pi^*\Omega - \varepsilon[E]$. Then the following are equivalent:*

(1) $\mathrm{Bl}_p X$ *admits an extremal metric in* Ω_ε *for all* $0 < \varepsilon \ll 1$*;*
(2) $(\mathrm{Bl}_p X, \Omega_\varepsilon)$ *is relatively K-stable for all* $0 < \varepsilon \ll 1$*.*

Moreover, the relative K-stability criterion is an explicit finite dimensional condition.

5. Constructing metrics on the blowup

We now start our construction. On the cscK central fibre in the family (4.1), we will use Székelyhidi's construction that we outlined in the previous section. This gives in particular a one-parameter family ω_ε of symplectic forms on the underlying smooth manifold of the blowup, which we denote $\mathrm{Bl}_p M$. Our goal is then to construct a good lift

$$\Psi_\varepsilon : B \to \mathcal{J}(\mathrm{Bl}_p M, \omega_\varepsilon)$$

of the map Ψ to the blowup, so that the scalar curvature of the almost complex structures land in the space of holomorphy potentials for the central fibre.

One advantage of working with the symplectic manifold $(\mathrm{Bl}_p M, \omega_\varepsilon)$ is that even vector fields that do not correspond to holomorphy potentials on the non-zero fibres have a natural lift now – we just use the lift on the central fibre. That is, for any function $h \in \overline{\mathfrak{h}}$ of average 0 with respect to ω, we have a naturally defined lift h_ε which is the Hamiltonian of average 0 with respect to ω_ε of the lift of the corresponding vector field to $\mathrm{Bl}_p X_0$. The only difference in the non-zero fibres is that not all of these functions are holomorphy potentials – they are Hamiltonians on the blowup, but they do not all produce a holomorphic vector field. We will let $\overline{\mathfrak{h}}_\varepsilon$ denote the space of such lifted potentials, with respect to ω_ε.

The goal now is to show that we can solve

$$S(\omega_\varepsilon, J_{\varepsilon,b}) \in \overline{\mathfrak{h}}_\varepsilon,$$

where $J_{\varepsilon,b} = \Phi_\varepsilon(b)$. It is only after solving this equation that we analyse the relation to K-stability. To do so, we perform the same construction as on the central fibre, on the non-zero fibres. This creates an initial $\omega_{\varepsilon,b}$, which in general is different from ω_ε. Thus to be able to take the symplectic point of view of having a fixed symplectic manifold (for each ε), we need to apply Moser's trick again. This gives a diffeomorphism $f_{\varepsilon,b}$ of M, giving a Kähler isomorphism

$$(\mathrm{Bl}_p M, J_b, \omega_{\varepsilon,b}) \cong (\mathrm{Bl}_p M, f^*_{\varepsilon,b} J_b, \omega_\varepsilon).$$

At this stage, we have put ourselves in the symplectic framework, but the Kähler structure $(\mathrm{Bl}_p M, f^*_{\varepsilon,b} J_b, \omega_\varepsilon)$ does not solve any special equation. To land in the space of holomorphy potentials on the central fibre, we have to perturb.

An important fact in the deformation theory, due to Székelyhidi ([**39**]) and Brönnle ([**7**]), is that we can ensure that the initial map $\Psi : B \to \mathcal{J}(M, \omega)$ before the blowup has scalar curvature that lands in $\overline{\mathfrak{h}}$, i.e. we have

$$S(\omega, J_b) \in \overline{\mathfrak{h}}$$

for all $b \in B$. This is what we wish to obtain also on the blowup.

Let $H_b = S(\omega, J_b) - \hat{S}$ which lies in $\overline{\mathfrak{h}}$, and let $H_{\varepsilon,b}$ be the corresponding lift. Then similarly to Lemma 3.3 we have the following.

LEMMA 3.6. *For all $\delta < 0$, there exists a $C > 0$ such that for $b \in B$ and for all sufficiently small positive ε,*

$$\|S(\omega_\varepsilon, J_b) - H_{\varepsilon,b} - \hat{S}\|_{C^{k,\alpha}_\delta} \leq C r_\varepsilon^{-\delta}$$

The next step is to perturb so that the scalar curvature lies in $\overline{\mathfrak{h}}_\varepsilon$. Since for each ε we are keeping the symplectic form fixed, we have to change the almost complex structure. Again, this goes through the Moser trick: if M is a symplectic manifold with almost complex structure J and f is a function, we can flow through the vector field dual to df to produce a symplectomorphism $F_f : M \to M$. We then define a new almost complex structure $F_f^* J$. The linearisation of the map $f \mapsto F_f^* J$ is the Lichnerowicz operator (with no gradient scalar curvature term).

Again, we can establish bounds for the linearisation in the whole family over B building on the bounds already established in the case of the central fibre in Proposition 3.4.

PROPOSITION 3.7. *Assume the dimension is at least* 3*. Then for $\delta \in (4-2n, 0)$, the operator $C^{k+4,\alpha}_\delta \times \mathfrak{h} \times \mathbb{R} \to C^{k,\alpha}_{\delta-4}$ given by*

$$(\phi, h, c) \mapsto L_\varepsilon(\phi) - h_\varepsilon - c$$

is surjective with right inverse Q_ε satisfying the uniform estimate

$$\|Q_\varepsilon\|_{C^{k,\alpha}_{\delta-4} \to C^{k+4,\alpha}_\delta} \leq C$$

for some $C > 0$.

There are similar bounds in dimension 2, but then they blow up with ε. Again, this is also the case in the work of Székelyhidi and causes manageable complications that we will not go into here.

The proof now follows similar steps to that of the proof of Székelyhidi. In the end we end up with a complex structure $J_{\varepsilon,b}$ such that

$$S(\omega_\varepsilon, J_{\varepsilon,b}) \in \overline{\mathfrak{h}}_\varepsilon.$$

We recall again that $\overline{\mathfrak{h}}_\varepsilon$ corresponds to holomorphy potentials on the *central fibre*, and so we have not necessarily solved the extremal equation on the non-zero fibres yet. We turn to understanding when we have done this now.

6. Relating the construction to K-stability

So far, we have solved

$$S(\omega_\varepsilon, J_{\varepsilon,b}) \in \overline{\mathfrak{h}}_\varepsilon.$$

It is only on the central fibre that all the functions in $\overline{\mathfrak{h}}_\varepsilon$ correspond to holomorphy potentials, and so it is only on the central fibre that we have guaranteed to solve the extremal equation. The holomorphy potentials on the non-central fibres lie in a proper subspace of $\overline{\mathfrak{h}}$.

The vector space the ball B lies in has a linear action of a torus T, which is a maximal torus of the reduced automorphism group of the cscK/extremal central fibre. We want to understand the condition that we can find, in a given orbit, a point such that the scalar curvature actually lies in the space of holomorphy potentials (or is constant, if we are solving the cscK problem).

The key to achieving this is that we can view this as looking for a zero, or a critical point, of a certain moment map. The reason we can do this is that we can view the map

$$b \mapsto J_{\varepsilon,b}$$

as giving a symplectic embedding into the space $\mathcal{J}(\mathrm{Bl}_p M, \omega_\varepsilon)$ of ω_ε-compatible almost complex structures on $\mathrm{Bl}_p M$.

Proposition 3.8. *The image of the map $B \to \mathcal{J}(\mathrm{Bl}_p M, \omega_\varepsilon)$ given by*

$$b \mapsto J_{\varepsilon,b}$$

is a symplectic submanifold.

The scalar curvature is a moment map for the action of the Hamiltonian symplectomorphism group on $\mathcal{J}(\mathrm{Bl}_p M, \omega_\varepsilon)$, hence its composition with the orthogonal projection to $\overline{\mathfrak{h}}$ is a moment map for the restriction to T of this action. This then also holds on the symplectic submanifold given by the image of the embedding $B \to \mathcal{J}(\mathrm{Bl}_p M, \omega_\varepsilon)$. Thus we have managed to put ourselves in the position that solving the cscK/extremal equation becomes a finite dimensional moment map problem, which can then be related to a GIT notion of stability/relative stability.

Now, for any element u in the Lie algebra of the torus one can produce a test configuration for X_b, if the element is rational (and a so-called $\mathbb{R}$ test configuration if the element is irrational – we will not go further into this point). This also gives a test configuration for $\mathrm{Bl}_p X_b$. Moreover, the value of the corresponding hamiltonian function at the limit point that b goes to under the action on B generated by u is the Donaldson–Futaki invariant of this test configuration. These ideas were shown by Székelyhidi in [**39**].

Thus assuming relative K-stability, we get a particular sign for all the Hamiltonian functions at limit points of b. A careful analysis shows that the assumption that all the Hamiltonian functions are negative at the limit point b allows one to produce another point in the same orbit as b where the value of all the Hamiltonians orthogonal to those that are holomorphy potentials vanishes. This uses a rather general framework that Dervan developed in [**16**]. In particular, assuming relative K-stability, we get the scalar curvature lies in the space of holomorphy potentials at b. Thus we have produced an extremal metric under the assumption of relative K-stability, completing the proof of the main result.

We end the lecture by noting that we also obtain explicit expansions that give the exact criterion for stability in this case. This follows from an explicit expansion of the Donaldson–Futaki invariant, using very similar ideas to the analogous statements in Székelyhidi's work on blowups ([**40, 42**]).

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