Combinatorics, Lecture 10, 2022/6/21

§.

Def. A <u>probability space</u> is a pair (Ω, P), where Ω is a finite set and $P: 2^{\Omega} \to [0,1]$ is a function satisfying the following conditions:

(1) $P(\phi) = 0$, (2) $P(\Omega) = 1$,

(3) $P(A \cup B) = P(A) + P(B)$ for disjoint sets $A, B \subseteq \Omega$.

Def. A subset A of Ω is called an <u>event</u>, and $P(A) = \sum_{w \in A} P(\{w\})$.

A <u>random variable</u> a function $X: \Omega \to R$.

Def. The <u>expectation</u> of a random variable X is $\mathbb{E}[X] = \sum_{w \in \Omega} P(\{w\}) \cdot X(w)$.

<u>Fact 1</u>. (Union bound) $\forall A, B \subseteq \Omega$,

$$P(A \cup B) \leq P(A) + P(B)$$

<u>Fact 2</u>. (The linearity of expectation). For any random variables X and Y, we have

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

§1. Union bound

Def. Recall. $R(s,t)$ (Ramsey number)

Thm 1. Let n, s satisfy $\binom{n}{s} 2^{1-\binom{s}{2}} < 1$.

Then $R(s,s) > n$.

Pf. We need to find a 2-edge-coloring of K_n such that it has NO monochromatic clique K_s.

Consider a random 2-edge-coloring c of K_n, where each edge of K_n is colored by red or blue, each with probability $\frac{1}{2}$ and independent of each other.

(This is the same as the following: Let Φ be the family of all 2-edge-colorings of K_n, and let $c \in \Phi$ be obtained by chosing uniformly at random.)

Let B be event that this coloring c has NO monochromatic K_s. We want to show $\boxed{P(B) > 0}$.

Consider the complement event $A = \Phi \setminus B$, i.e. A is the event that c has a monochromatic K_s.

For $S \in \binom{[n]}{s}$, Let A_S be the event that

S forms a monochromatic K_s for c.

$$\Rightarrow \quad A = \bigcup_{S \in \binom{[n]}{s}} A_S \quad \text{and} \quad P(A_S) = 2^{1-\binom{s}{2}}.$$

By the union bound,

$$P(A) = P\Big(\bigcup_{S \in \binom{[n]}{s}} A_S\Big) \leq \sum_{S \in \binom{[n]}{s}} P(A_S)$$

$$= \binom{n}{s} 2^{1-\binom{s}{2}} < 1.$$

i.e. $P(B) > 0$. $\boxed{}$

Corollary. $R(s,s) > \frac{1}{e\sqrt{2}} s \, 2^{s/2}$.

Pf. (Exercise) Choose a proper $n = n(s)$ so that the condition of Thm 1 satisfies. $\boxed{}$

Def. The random graph $G(n,p)$ for some $p \in (0,1)$ is a graph with vertex set $[n]$, where each of the potential $\binom{n}{2}$ edges appears with probability p, independent of each other. (Often, we consider $p = \frac{1}{2}$).

- Let A be an event / a graphic property that we are interested in. Let

$$P(A) = P(\, G(n,\tfrac{1}{2}) \text{ satisfies the property } A \,)$$

a function on n

$$= \frac{\#\text{ of } n\text{-vtx graphs on } [n] \text{ satisfying } A}{2^{\binom{n}{2}}}$$

Def. We say $G(n,\frac{1}{2})$ <u>almost surely</u> satisfies a property A, if $\lim_{n\to\infty} P(A) = 1$

Thm 2. $G(n,\frac{1}{2})$ almost surely is non-bipartite.

Pf: Let A be the event that $G(n,\frac{1}{2})$ is bipartite.

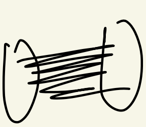
So we need to show $\lim_{n\to\infty} P(A) = 0$.

Fixed a subset $U \subseteq [n]$, let A_U be the event that $G(n,\frac{1}{2})$ is bipartite with bipartition $(U, [n]\setminus U)$. So $A = \bigcup_{U \subseteq [n]} A_U$.

We have

$$P(A_U) = \frac{\#\text{ graphs on } [n] \text{ satisfying } A_U}{2^{\binom{n}{2}}}$$

$$= \frac{2^{|U|\cdot(n-|U|)}}{2^{\binom{n}{2}}}$$



U $\quad$ $[n]\setminus U$

$|U|(n-|U|)$

$$\le \frac{2^{n^2/4}}{2^{(n^2-n)/2}} = 2^{-\frac{n^2}{4}+\frac{n}{2}}.$$

By union bound,

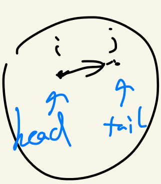
$$0 \le P(A) \le \sum_{U \subseteq [n]} P(A_U) \le 2^n \cdot 2^{-\frac{n^2}{4}+\frac{n}{2}}$$

$$= 2^{-\frac{n^2}{4}+\frac{3}{2}n}$$

$$\Rightarrow \lim_{n\to\infty} P(A) = 0. \qquad \square$$

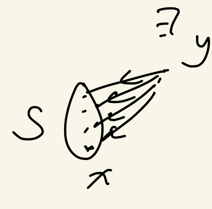
Def. Given a probability space (Ω, P), events $A_1, A_2, \dots, A_k$ are <u>independent</u>, if for any $I \subseteq [k]$, we have $P\left(\bigcap_{i\in I} A_i\right) = \prod_{i\in I} P(A_i)$.

Def A <u>tournament</u> on n vertices is a directed graph obtained from K_n by assigning a direction to each edge.



An <u>arc</u> is an edge in a directed graph.

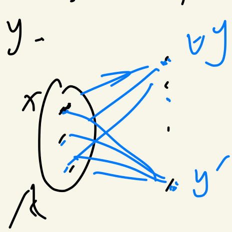
Def A tournament satisfies <u>the property S_k</u> if for any subset S of size k, there exists a vertex $y \notin S$ such that $y \to x$ for any $x \in S$.



Thm 3. For $\forall k$, if $\boxed{\binom{n}{k}\left(1-\frac{1}{2^k}\right)^{n-k} < 1}$, then there exists a tournament on n vertices satisfying the property S_k.

pf: We consider a random tournament T on $[n]$, where for any pair $\{i,j\}$, the arc $i \to j$ occurs with probability $\frac{1}{2}$, independent of each other.

Let B be the event that T does not satisfy property S_k. For fixed $A \in \binom{[n]}{k}$, let B_A be the event that the k-set A "fails", i.e. $\forall y \in [n]\setminus A$, $\exists x \in A$ with $x \to y$.



$$\Longrightarrow B = \bigcup_{A \in \binom{[n]}{k}} B_A .$$

For $y \in [n]\setminus A$, let $B_{A,y}$ be the event that $\exists x \in A$ with $x \to y$.

$$P(B_{A,y}) = 1 - \left(\tfrac{1}{2}\right)^k .$$

Note that $B_A = \bigcap_{y \in [n]\setminus A} B_{A,y}$ and events

$B_{A,y}$'s are independent (as the k arcs between y and A are distinct for each y.)

$$\Rightarrow \quad P(B_A) = \prod_{y \in [n] \setminus A} P(B_{A,y})$$

$$= \left(1 - \frac{1}{2^k}\right)^{n-k}.$$

$$\Rightarrow \quad P(B) \leq \sum_{A \in \binom{[n]}{k}} P(B_A) \leq \binom{n}{k} \cdot \left(1 - \frac{1}{2^k}\right)^{n-k}$$

$$< 1$$

$\boxtimes$

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§2. The linearity of expectations

Facts. For two random variables X, Y,

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y], \quad P(X \geq \mathbb{E}[X]) > 0$$

and $P(X \leq \mathbb{E}[X]) > 0$.

Def. A set A is sum-free, if for any $x, y \in A$, $x+y \notin A$. (That is, $x+y=z$ has NO solution in A)

Examples $\{\lceil \frac{n}{2} \rceil + 1, \lceil \frac{n}{2} \rceil + 2, \ldots, n\}$ is sum-free.

$\{$ all odd integers in $[n]\}$ is sum-free.

Exercise. The maximum size of a sum-free subset in $[n]$ is $\lceil \frac{n}{2} \rceil$.

Thm 4. Let A be a finite set of non-zero integers. There exists a sum-free subset $B \subseteq A$ with $|B| \geq \frac{|A|}{3}$.

Pf: We choose a large prime p such that $p > |a|$ for $\forall a \in A$. Consider $\mathbb{Z}_p = \{0, 1, \cdots, p-1\}$ and $\mathbb{Z}_p^* = \{1, 2, \cdots, p-1\}$. Note that there exists a sum-free subset S under $\mathbb{Z}_p$ (mod p):

$$S = \left\{ \left\lceil \frac{p}{3} \right\rceil + 1, \left\lceil \frac{p}{3} \right\rceil + 2, \cdots, \left\lceil \frac{2p}{3} \right\rceil \right\}.$$

Claim. For $x \in \mathbb{Z}_p^*$, $A_x = \{ a \in A : ax \pmod p \in S \}$ is sum-free subset of A.

Pf. Suppose $a, b, c \in A_x$ with $a + b = c$.

Then $ax \pmod p$, $bx \pmod p$, $cx \pmod p \in S$.

$$\& \quad ax + bx \equiv cx \pmod p,$$

a contradiction to that S is sum-free under $\mathbb{Z}_p$. $\square$

Next, we turn to find an $x \in \mathbb{Z}_p^*$ with $|A_x| \geq |A|/3$. We will find such x by choosing $x \in \mathbb{Z}_p^*$ uniformly at random.

Compute $\mathbb{E}[|A_x|]$,

where $|A_x| = \sum_{a \in A} \mathbb{1}_{\{ax \bmod p \in S\}}$.

So $\mathbb{E}[|A_x|] = \mathbb{E}\left[\sum_{a \in A} \mathbb{1}_{\{ax \bmod p \in S\}}\right]$

$$= \sum_{a \in A} \mathbb{E}[\mathbb{1}_{\{ax (\bmod p) \in S\}}]$$

$$= \sum_{a \in A} \mathbb{P}(ax \bmod p \in S).$$

We observe that for fixed $a \in A$ $\Rightarrow (a \in \mathbb{Z}_p^*)$

$$\{ax : x \in \mathbb{Z}_p^*\} = \mathbb{Z}_p^*.$$

Then $\mathbb{P}(ax \bmod p \in S) = \frac{|S|}{|\mathbb{Z}_p^*|} \geq \frac{1}{3}$.

$\Rightarrow \mathbb{E}[|A_x|] \geq \frac{1}{3}|A|$.

As $\mathbb{P}(|A_x| \geq \mathbb{E}[|A_x|]) > 0$, there exists some A_x with $|A_x| \geq \mathbb{E}[|A_x|] \geq \frac{1}{3}|A|$. $\boxed{\Sigma}$

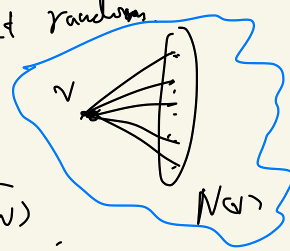
Def. Let $\alpha(G)$ be the maximum size of an independent set in a graph G.

Thm 5. For any graph G, $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d(v)+1}$ where $d(v)$ denotes the degree of the vertex $v \in V(G)$.

Pf: Let S_n be the family of all permutations $\pi : [n] \to [n]$. Let $V(G) = [n]$.

For a given permutation π, we say a vertex $v \in V(G)$ is <u>π-good</u> if $\pi(v)$ is the smallest value over the vertices $\{v\} \cup N_G(v)$.

Let $\pi \in S_n$ be chosen uniformly at random.



Then $\forall\ v \in V(G)$

$$\mathbb{P}(v \text{ is } \pi\text{-good}) = \frac{1}{1+d(v)}.$$

<u>Claim</u>. All π-good vertices form an independent set.

Pf: Suppose $\exists$ two π-good vertices i, j with $ij \in E(G)$.

Let $\pi(i) < \pi(j)$. $\Rightarrow$ j is NOT π-good, <u>a contradiction.</u> $\boxtimes$

Compute $\mathbb{E}\left[\,\left|\{\text{all } \pi\text{-good vertices}\}\right|\,\right]$

$$= \sum_{v \in V(G)} \mathbb{E}\left[\mathbb{1}_{\{v \text{ is } \pi\text{-good}\}}\right]$$

$$= \sum_{v \in V(G)} \mathbb{P}(v \text{ is } \pi\text{-good})$$

$$= \sum_{v \in V(G)} \frac{1}{1+d(v)}.$$

Then there exists some $\pi \in S_n$ with

$$\alpha(G) \geq \left| \{ \text{all } \pi\text{-good vertices} \} \right| \geq \mathbb{E}\left[\left| \{ \pi\text{-good vertices} \} \right| \right]$$

$$\geq \sum_{v \in V(G)} \frac{1}{1 + d(v)} . \qquad \boxtimes$$

<u>Corollary</u>. For any graph G on n vertices and m edges, we have $\alpha(G) \geq \frac{n^2}{2m+n}$. $\boxtimes$

<u>Pf.</u> Exercise.

<u>Turán's Thm</u> (Approximate version) If G has n vertices and is K_{r+1}-free, then $e(G) \leq \frac{r-1}{2r} n^2$.

<u>2^{nd} pf.</u> Using previous result. $\boxtimes$

§ 3. The deletion method

Earlier, we often define an approximate probability space and show that the random events occurs with positive probability.

Now we extend this idea, and consider situations where the random events does not always have the desired property, and may have very <u>few "blemishes"</u>

The point that we want to make here is that after deleting all blemishes, we will obtain the desired property

Recall: If $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$.

$$\Longrightarrow \quad R(k,k) > \frac{1}{e\sqrt{2}} k 2^{k/2}.$$

Thm 6. For any integer n, we have

$$R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}.$$

Pf. Consider a random 2-edge-coloring of K_n.

For $A \in \binom{[n]}{k}$, let X_A be the indicator random variable of the event that A induces a monochromatic K_k. Then $\mathbb{E}[X_A] = 2^{1-\binom{k}{2}}$.

Let $X = \sum_{A \in \binom{[n]}{k}} X_A$ be the number of monochromatic K_k's. Then

$$\mathbb{E}[X] = \sum_{A \in \binom{[n]}{k}} \mathbb{E}[X_A] = \binom{n}{k} 2^{1-\binom{k}{2}}.$$

So there exists a 2-edge-coloring of K_n whose number of monochromatic K_k is at most X

$$\mathbb{E}[X] = \binom{n}{k} 2^{1-\binom{k}{2}}.$$

Next, we remove one vertex from each monochromatic K_k. This will delet at most $X \leq \binom{n}{k} 2^{1-\binom{k}{2}}$ vertices, and will destroy all monochromatic K_k. So it remains at least $n - \binom{n}{k} 2^{1-\binom{k}{2}}$ vertices with no monochromatic K_k.

$$\Rightarrow \quad R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}} \qquad \square$$

Corollary. $R(k,k) > \frac{1}{e}(1+o(1))\, k\, 2^{k/2}$

Pf. Exercise. $\square$