2020-9-29 Kähler geometry

10/9 Friday $\to$ 10/11 Sunday

10/23 Friday $\to$ 10/18 Sunday

All the course will be online until the end.

Twisted Kähler-Einstein metric.

Let M be a compact Kähler manifold of dim m.

Let ω be a fixed Kähler form.

$[\omega]$ Kähler class (fixed).

$\omega_\varphi = \omega + i\partial\bar\partial\varphi$, $\varphi \in C^\infty(M)$.

another Kähler form in $[\omega]$.

Let $F \in C^\infty(M)$.

Solve the following complex Monge-Ampère equation.

$$\frac{\det(g_{i\bar j} + \varphi_{i\bar j})}{\det(g_{i\bar j})} = e^{-\lambda\varphi + F} \qquad \varphi_{i\bar j} = \frac{\partial^2\varphi}{\partial z^i \partial \bar z^j}$$

$\lambda = -1, 0$ or 1.

on any compact Kähler manifold M.

Theorem (Yau $\lambda = 0, -1$, Aubin $\lambda = -1$)
1976 – 1977

$$\overbrace{\omega \wedge \cdots \wedge \omega}^{n} = \omega^n$$

On any compact Kähler manifold for any $F \in C^\infty(M)$ $\left(\text{with } \int_M e^F \omega^n = \int_M \omega^n \text{ for } \lambda = 0 \right)$

there exists a unique solution $\varphi \in C^\infty(M)$ of the equation for $\lambda = -1$ and $\lambda = 0$.

(requiring $\int \varphi\, \omega^n = 0$ for $\lambda = 0$).

For $\lambda = 1$, there is no general existence result.

(Except for some special situation for Fano manifold related to Yau-Tian-Donaldson conjecture.)

The equation   is equivalent to solving the following problem.

Given a closed $(1,1)$-form α representing $c_1(M) - \lambda[\omega]$, does there exist a Kähler form $\omega_\varphi \in [\omega]$ such that

$$\underline{Ric(\omega_\varphi)} - \lambda \omega_\varphi = \alpha \quad ?$$

$$\cdots\cdots (1)$$

To see this equation, let $F \in C^\infty(M)$ satisfying

$$Ric(\omega) - \lambda\omega = \alpha + i\partial\bar\partial F$$

$$\cdots\cdots (2)$$

$(2) - (1)$ gives $\qquad \underline{\det(g_{i\bar j} + \varphi_{i\bar j})}\ \underline{dz^1 \wedge \cdots \wedge dz^{\bar n}}$

$$i\partial\bar\partial \log \frac{\omega_\varphi^n}{\omega^n} = i\partial\bar\partial(-\lambda\varphi + F)$$

$\underline{\det(a_{ij})}$

$$\therefore \frac{\omega_\varphi^n}{\omega^n} = e^{-\lambda\varphi + F + \text{const}}$$

F

$$= e^{-\lambda\varphi + F}$$

Cor (Yau, Aubin)

for $\lambda = -1, 0$, the twisted K-E equation can be always solved.

Cor (Calabi conjecture)

On any compact Kähler mfd with fixed Kähler class, given any real closed (1,1)-form α representing $c_1(M)$, there exists a unique Kähler metric ω in the fixed Kähler class s.t.

$$Ric(\omega) = \alpha.$$

<u>Rem</u> $\alpha = 0$, $\lambda = 1$ $\rightsquigarrow$ Yau-Tian-Donaldson conj solved for Fano mfd.

<u>Theorem</u> (twisted KE case $\lambda = 1$, Berman-Boucksom-Jonsson (2015/2020 version algebraic case, Kewei Zhang 2020 transcendental case)

Let (M, ω) be a compact Kähler manifold and α be a closed 2-form representing

$c_1(M) - [\omega]$. Assume $\alpha \geq 0$

(so that $c_1(M) = [\omega] + [\alpha] > 0$
so M Fano)

Then we have

(1) If $\delta([\omega]) > 1$ then $\exists \omega_\varphi \in [\omega]$
such that $Ric(\omega_\varphi) = \omega_\alpha + \varphi$

(2) If $\exists \omega_\varphi \in [\omega]$ (resp. $\exists! \omega_\varphi \in [\omega]$)
s.t. $Ric(\omega_\varphi) = \omega_\varphi + \alpha$ then
$\delta([\omega]) \geq 1$ (resp $\delta([\omega_\varphi]) > 1$).

The def of $\delta([\omega])$ is as follows.

$L \in N^1(X)_{\mathbb{R}}$ be a big $\mathbb{R}$-line bundle
in Néron-Severi space

$$\delta(L) = \inf \frac{A(F)}{S_L(F)} \qquad \sum(1-a_i)E_i$$

$A(F)$ = log discrepancy of F $\qquad \int \frac{dx}{x^{a_i}}$

$$S_L(F) = \frac{1}{vol(L)} \int_0^\infty vol(L - tF)$$

$$\underline{K_Y} + \pi_*^{-1}(F) + E = \pi^*(K_X + F) + \sum a_i E_i$$

min a_i = log discrepancy of F.

Basics of integration in Riemannian and Kählerian manifolds.

Let (M, g) be a compact oriented manifold without boundary. $\dim_{\mathbb{R}} M = n$.

$$dV_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$$

in coordinate nbds.

exercise: This is independent of local coordinates.

dV_g gives a measure on (M, g) "volume element".

$$X = X^i \frac{\partial}{\partial x^i}$$

$$\underbrace{d\big(\overbrace{i(X)\, dV_g}^{(n-1)\text{ form}}\big)}_{n\text{-form}} =: \operatorname{div} X \cdot dV_g$$

Given p there is a coord system s.t. $\Gamma^k_{ij}(p) = 0$ at p. $g_{ij} = \delta_{ij}$ at p, $dg|_p = 0$ "normal coord"

$$\operatorname{div} X = \nabla_i X^i$$

(exercise: use normal coordinates).

By Stokes theorem

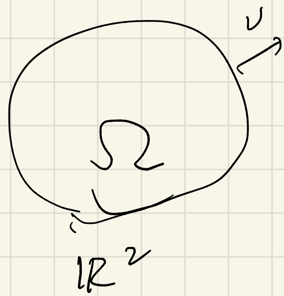
"div X" is called the divergence of X.

$$\int_M \operatorname{div}(X)\, dV_g = 0.$$

$$\| $$

$$\int_M d\big(i(X)\, dV_g\big)$$

"divergence theorem"



$$\left(\int_\Omega \sum_i \frac{\partial X^i}{\partial x^i} = \int_{\partial\Omega} \langle \nu, X \rangle \, d\sigma \right)$$

Application

$\langle df, T \rangle$ ← vector field

(i) $$\int_M T^i \cdot \nabla_i f \, dV_g$$

$\operatorname{div}(fT)$

$$= \underbrace{\int_M \nabla_i (T^i f) \, dV_g}_{=0} - \int_M (\nabla_i T^i) \cdot f \, dV_g$$

$$= - \int_M (\nabla_i T^i) \cdot f \, dV_g.$$

"integration by parts"

(2) $\Delta f = \nabla^i \nabla_i f = \nabla_i \nabla^i f$

$$\underbrace{\nabla^i \nabla_i f}_{} \overset{\text{"}}{} \; g^{ij} \nabla_i \nabla_j f = g^{ij}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k}\right)$$

$$\nabla_j f = \frac{\partial f}{\partial x^j}$$

$f \in C^\infty(M)$.

$\nabla_i \nabla_j f = \nabla_j \nabla_i f$

$\therefore \quad \Delta f = \operatorname{div}(\# df)$

$\therefore \quad \int_M \Delta f \, dV_g = 0.$

(3) Conversely if $\int_M u \, dV_g = 0$

then $\exists f \in C^\infty(M)$ s.t. $\Delta f = u$.

(Exercise : Use Hodge ~~theory~~)

<u>Kähler case:</u>

$$\frac{\omega^m}{m!} = \det(g_{i\bar{j}}) \, i^m dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^m \wedge d\bar{z}^m$$

$$= dV_g$$

But we usually omit $m!$

and consider ω^m as the volume form.

$$\Delta_d = d^* d + d d^* \quad \text{on Riemannian}$$

$$\Delta_{\bar\partial} = \bar\partial^* \bar\partial + \bar\partial \bar\partial^* \quad \text{on Kähler}$$

$$\shortparallel$$

$$\Delta_\partial = \partial^* \partial + \partial \partial^*$$

Also $\Delta_d = \Delta_{\bar\partial} + \Delta_\partial = 2 \Delta_{\bar\partial}$

$$\Delta_{\bar\partial} f = g^{i\bar j} \nabla_i \nabla_{\bar j} f = g^{i\bar j} \frac{\partial^2 f}{\partial z^i \partial \bar z^j}$$

$$= \frac{1}{2} \Delta_d f$$

More convenient to write $\Delta = \Delta_{\bar\partial}$

$$\int_M u \cdot \nabla_i X^i \, \omega^m = - \int \nabla_i u \cdot X^i \, \omega^m$$

$$= - \int X(u) \, \omega^m.$$

$\partial\bar{\partial}u = 0 \implies \Delta u = 0 \implies u = \text{const}$

$$0 = \int_M u \cdot \Delta u = -\int \nabla_i u \cdot \nabla^i u \ \omega^n$$

$$= -\int_M |\bar{\partial} u|^2 \omega^n$$

$\bar{\partial} u = 0 \qquad u = \text{const}.$

Any harmonic function on a compact Riem mfd without bdry is constant