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Khovanov homology for virtual knots with arbitrary coefficients

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Abstract. The Khovanov homology theory over an arbitrary coefficient ring is extended to the case of virtual knots. We introduce a complex which is well-defined in the virtual case and is homotopy equivalent to the original Khovanov complex in the classical case. Unlike Khovanov's original construction, our definition of the complex does not use any additional prescription of signs to the edges of a cube. Moreover, our method enables us to construct a Khovanov homology theory for 'twisted virtual knots' in the sense of Bourgoin and Viro (including knots in three-dimensional projective space). We generalize a number of results of Khovanov homology theory (the Wehrli complex, minimality problems, Frobenius extensions) to virtual knots with non-orientable atoms.

§ 1. Introduction

1.1. Virtual knots. Virtual knots were introduced in mid-1990s by Kauffman [1]. A *virtual diagram* is a planar four-valent graph with the following additional structure: each crossing is said to be either classical (with a pair of opposite edges being indicated as forming an overcrossing at this point) or virtual (such crossings are just marked by a circle). Classical knots, whose original definition is topological, may also be treated as combinatorial objects: equivalence classes of planar diagrams with respect to the Reidemeister moves. We introduce the more general notion of a *virtual link*: an equivalence class of virtual diagrams with respect to generalized Reidemeister moves. The generalized Reidemeister moves consist of the classical Reidemeister moves (see [2], for example) and detour moves. To make a detour move, we delete an arc AB containing only virtual crossings and add another arc connecting A and B. All crossings and self-crossings belonging to the new arc are declared to be virtual (see Fig. 1).

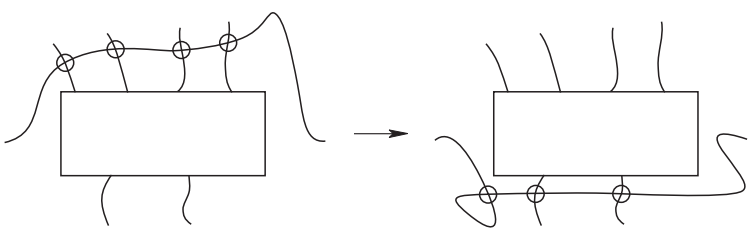

Figure 1. The detour move

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In what follows we mainly use the combinatorial definition of virtual knots. The topological structure of virtual knots will also be described below.

A *virtual crossing* is neither an overcrossing nor an undercrossing: we imagine it as being formed by two disjoint arcs which cannot be shown disjoint for reasons of dimension. Geometrically, virtual knots are obtained from knots in the 'thickened surfaces' $S_g \times \mathbb{R}$, where S_g is a handlebody (such knots were studied in [3]). The handlebody need not be embedded in three-dimensional space.

If we map the oriented surface S_g into $\mathbb{R}^2$, then the thickened surface $S_g \times \mathbb{R}^1$ is mapped into $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}^1$. We assume that the map is orientation-preserving near all classical crossings. Then the classical crossings are mapped to classical crossings while the virtual crossings appear as artefacts of the projection. (Different points of the curve in S_g may be projected to the same point of the planar diagram.) These crossings are 'actually absent', and an arc which contains them may be moved to any other place. This enables us to give a formal definition of virtual knots in terms of abstract diagrams (called *Gauss diagrams*). Classical knots are knots in the thickened sphere. This is consistent with the fact that classical links appear as equivalence classes of link diagrams in S^2 (or $\mathbb{R}^2$) with respect to the classical Reidemeister moves. Given a virtual diagram on the plane, we can construct a diagram on the handlebody. Adding or removing 'empty' handles (disjoint from the diagram), we get the same virtual knot. An important theorem (see [4], for example) asserts that virtual knots coincide with knots in thickened surfaces up to the isotopies and homeomorphisms of the thickened surface $S_g \times \mathbb{R}^1$ that preserve the coordinate along $\mathbb{R}^1$, and up to stabilization/destabilization of the surface. The key fact is given by a theorem of Kuperberg [5]: every class of virtual knots has a unique representative with a minimal number of handles. It follows that classical knot theory is a part of the theory of virtual knots: if two classical knots are equivalent as virtual knots, then they are equivalent as classical knots (see [2]).

1.2. Khovanov homology. A categorification of the Jones polynomial was suggested by Khovanov (see [6], as well as [7], [8]). It is based on the passage from Laurent polynomials to homology groups of complexes of bigraded vector spaces.[1] The paper [6] defines a bigraded chain complex for any given link diagram. All the homology groups of this complex are knot invariants, and the Euler characteristic of the complex coincides with the Jones polynomial (see below).

We shall consider bigraded complexes of the form $\mathcal{C} = \bigoplus_{i,j} \mathcal{C}^{i,j}$ with *height* i and *grading* j. The differential ∂ of the complex preserves the grading and increases the height by 1.

We define the *graded dimension* qdim of a graded vector space A by

$$\operatorname{qdim} A = \sum_j q^j \dim A^j,$$

where A^j is the subspace of A generated by all vectors of grading j. The spaces $\mathcal{C}^i$ of chains of height i in the complex $\mathcal{C}$ are examples of graded vector spaces.

[1] In what follows we use the term *vector space*, although our construction applies to modules over an arbitrary commutative ring.

It would be more appropriate to use the term *cohomology* since the differential increases the height by 1. However, we use the term *Khovanov homology* and speak of 'chains', 'cycles', 'boundaries' instead of 'cochains', 'cocycles', 'coboundaries'.

The operation $\mathcal{C} \mapsto \mathcal{C}[k]\{l\}$ shifts the grading and height of a bigraded complex. It is defined by $\big(\mathcal{C}[k]\{l\}\big)^{i,j} = \mathcal{C}^{i-k,j-l}$. The *graded Euler characteristic* of the complex $\mathcal{C}^{i,j}$ is the alternating sum of the graded dimensions of the chain spaces or, equivalently, of the homology groups. For the chain spaces, we have

$$\sum_i (-1)^i \operatorname{qdim} \mathcal{C}^i = \sum_{i,j} (-1)^i q^j \dim \mathcal{C}^{i,j}.$$

1.3. The main result. The author [9], [10] has constructed a Khovanov homology theory for virtual links. More precisely, the Khovanov complex was defined for virtual link diagrams with orientable atoms. In the general case (including that of non-orientable atoms), the Khovanov complex was defined only over $\mathbb{Z}_2$, when the commutativity of the faces of the cube is equivalent to their anti-commutativity (this yields the basic fact that the square of the differential vanishes). An important role in the construction and study of the Khovanov homology for classical and virtual knots is played by the notion of an atom (see the definition in § 2.2), in particular, by its genus and orientability/non-orientability. Non-orientability was the main obstruction to constructing a general theory. For non-orientable atoms, we suggested two geometric constructions [10] that transform any virtual link to a virtual link with orientable atom. We also proved that the Khovanov homology groups of the resulting virtual knot (link) are invariants of the original one.

The main difficulty which is overcomed in this paper is in defining the differential for complexes corresponding to arbitrary virtual knots. Here one must consider many more cases than for the classical knots. The first key idea is to change the basis of the Frobenius algebra (which is the Khovanov homology of the unknot) as we pass from one crossing of the knot to another. The second key idea is to replace the usual tensor product (corresponding to several circles in a given state) by the *exterior product* of the corresponding graded spaces. This enables us to avoid the 'artificial' procedure of transforming the commutative cube into an anti-commutative one, as was done in [6], [10], [11].

For every virtual link diagram we define a bigraded complex whose homology groups are invariant under generalized Reidemeister moves. We mention some important properties of this construction.

1. The construction of the complex uses atoms. The complex is invariant under so-called virtualization. (See the definition on p. 973: virtualization changes the virtual knot diagram but preserves the corresponding atom and the Jones polynomial of the knot.) This is proved in Lemma 1.

2. There is a natural map from the set of 'twisted virtual knots' to the set of virtual knots modulo virtualization. Therefore our approach yields invariants of twisted virtual knots. The set of twisted virtual knots (knots in oriented thickenings of non-orientable two-dimensional surfaces up to stabilization) contains all knots in the punctured three-dimensional projective space. Hence our theory yields the Khovanov homology for knots in $\mathbb{R}P^3$.

3. For the coefficient field $\mathbb{Z}_2$, our complex coincides precisely with the complex constructed in [9].

4. For orientable atoms (in particular, for classical knots), the homology of our complex is the same as the homology of the complex constructed in [10].

5. The proof of invariance of the homology is local: it repeats the proof of invariance in the classical case (see [11], for example). The main difficulty is in defining the differential: choose the signs to make the cube anti-commutative. We overcome this difficulty by constructing a new complex which is homotopy equivalent to Khovanov's original complex.

1.4. Further properties of the Khovanov homology. A remarkable property of the Khovanov homology is its *functoriality*: every cobordism in $\mathbb{R}^3 \times I$ between links $K_1 \subset \mathbb{R}^3 \times \{0\}$ and $K_2 \subset \mathbb{R}^3 \times \{1\}$ generates a natural map $\mathrm{Kh}(K_1) \to \mathrm{Kh}(K_2)$ between the Khovanov homology groups, and this map is invariant (up to sign) under isotopies of the cobordism between K_1 and K_2.

The functoriality (up to sign) of the Khovanov homology theory was proved by Jacobsson [12]. A simpler proof for the wider topological theory was given in [13]. One can assume [13] that the Khovanov complex is built from complexes of elementary cobordisms, which in turn consist of elementary complexes: circles. This theory is a topological analogue of the algebraic theory described above: the objects of the category are unknots (sets of circles) and the morphisms are cobordisms.

There is another definition of the Khovanov homology [7]. Instead of counting the number of circles in each state, one considers two states of a fragment of the diagram (a neighbourhood of a crossing). For each fragment one defines an algebraic object: a *matrix factorization* with some *potential*. The matrix factorizations enable one to construct (by iterating tensor products and contractions) a formal chain complex whose homology is invariant with respect to Reidemeister moves. This remarkable construction is due to Khovanov and Rozansky. Later they developed it to categorify the HOMFLY polynomial. They write: 'The network does not even have to be planar, and does not need to be embedded anywhere. In our paper, however, all such diagrams are going to be planar' ([7], p. 32). This means that the construction of Khovanov and Rozansky (not only in the case of the usual Khovanov homology, also referred to as the $sl(2)$-homology, but even the general case which categorifies the HOMFLY polynomial) admits a generalization to virtual knots.

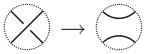
It is neither asserted nor proved in [7] that the $sl(2)$-theory of Khovanov and Rozansky is isomorphic to the usual Khovanov homology theory, where the chains and differentials are described by bifurcations of circles endowed with two-dimensional algebras. In this paper we explicitly construct the Khovanov homology for virtual knots, that is, we describe the chains and differentials by bifurcations of circles.

This explicit construction of the Khovanov homology leads to a number of important properties, including its invariance under virtualization. It follows that the Khovanov homology is restored from the 'atom' corresponding to the virtual knot diagram. This leads to the Khovanov homology theory for 'twisted virtual knots' (in the sense of Viro [14]), of which particular cases are knots in the three-dimensional projective space $\mathbb{R}P^3$.

We note that the Kauffman bracket for knots in $\mathbb{R}P^3$ was constructed in [15].

Moreover, this theory admit various generalizations, which are known for the usual Khovanov homology: Lee theory, spanning trees of Wehrli and Kofman–Champanerkar and so on.

§ 2. The Kauffman bracket and the Jones polynomial. Atoms. Twisted virtual knots

2.1. The Kauffman bracket. We consider an oriented virtual diagram L. Let $|L|$ be the non-oriented diagram obtained from L by forgetting the orientation. Each classical crossing of $|L|$ can be smoothed in one of two ways, A:  or B:  .

A *state* is a choice of smoothing for all classical crossings of the diagram. Each state determines a set of planar curves with only virtual crossings, that is, a virtual unlink diagram. We denote the number of classical crossings of L by n and enumerate them in an arbitrary way. We get a set of 2^n states which is in one-to-one correspondence with the set of vertices of an n-dimensional discrete cube $\{0,1\}^n$, where 0 and 1 correspond to the A-smoothing and B-smoothing respectively. This cube is called the *state cube* of the diagram. Two states are *adjacent* if they differ in one coordinate only. Any two adjacent states of the cube are connected by an edge. We orient each edge by the increase of the corresponding coordinate, that is, from the A-smoothing to the B-smoothing. The *height* of a vertex is the number of B-smoothings in the corresponding state.

Each edge determines a bifurcation of some state to an adjacent state. These states differ at one crossing only. There are 3 different types of edges: the number of circles in the given state may decrease by 1 (then we speak of a bifurcation of type $2 \to 1$ because two circles locally bifurcate into one at this crossing), increase by 1 (then we have a bifurcation of type $1 \to 2$), or remain unchanged (a bifurcation of type $1 \to 1$). The last of these is a singular case which can only occur for virtual knots. It leads to serious difficulties, but we shall overcome them in this paper.

In Khovanov's complex, bifurcations of type $2 \to 1$ and $1 \to 2$ correspond to the *partial differentials* ∂' (see § 4) which comprise the differential ∂ (see below). Each bifurcation $2 \to 1$ (resp. $1 \to 2$) corresponds to a multiplication m (resp. comultiplication Δ) in the vector spaces associated with the circles.

For every state s, let $\alpha(s)$ be the number of A-smoothings, $\beta(s) = n - \alpha(s)$ the number of B-smoothings, and $\gamma(s)$ the number of components of the unlink determined by s. Then the *Jones polynomial* is defined by

$$X(L) = (-a)^{-3w(L)} \sum_s a^{\alpha(s)-\beta(s)} (-a^2 - a^{-2})^{\gamma(s)-1}, \tag{1}$$

where $w(L)$ is the writhe of the oriented diagram L (the difference between the number of positive crossings and the number of negative crossings).

The non-normalized version $\sum_s a^{\alpha(s)-\beta(s)}(-a^2 - a^{-2})^{\gamma(s)-1}$ of the Jones polynomial is called the *Kauffman bracket*. Like the Jones polynomial, it is a Laurent polynomial in one variable. The Jones polynomial is invariant under generalized Reidemeister moves [1].

Changing the variable by writing $a = \sqrt{-q^{-1}}$, we get a modified version J of the Jones polynomial X. We also consider the polynomial $\hat{J} = J \cdot (q + q^{-1})$. To describe $\hat{J}$ in more detail, we consider an oriented virtual diagram L and the corresponding non-oriented virtual diagram $|L|$. We denote the number of positive (resp. negative) crossings of L by n_+ (resp. n_-) and put $n = n_+ + n_-$. Then we have

$$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle, \tag{2}$$

where $\langle L \rangle$ is the modified Kauffman bracket, which is defined axiomatically by the rules

$$\langle \bigcirc \rangle = (q + q^{-1}), \qquad \langle L \rangle = \langle L_A \rangle - q \langle L_B \rangle, \qquad \langle L \sqcup \bigcirc \rangle = (q + q^{-1}) \cdot \langle L \rangle.$$

Here L_A and L_B are the diagrams obtained from L by smoothing a fixed crossing in the ways A and B respectively.

In what follows we use $\hat{J}$ and call it the *Jones polynomial*.

The following description of $\hat{J}$ in terms of the state cube is conceptually important. Up to the normalizing factor $(-1)^{n_-} q^{n_+ - 2n_-}$, we have the (slightly modified) Kauffman bracket $\sum_s (-q)^{\beta(s)} (q+q^{-1})^{\gamma(s)}$. The sum is taken over all vertices of the cube, and each term is the product of $(-q)$ to the power equal to the *height of the vertex* and $q + q^{-1}$ to the power equal to the number of circles in the corresponding state.

Thus the polynomial is obtained by assigning the Laurent polynomial $q + q^{-1}$ to each circle and taking the sum of the products of these Laurent polynomials with coefficients $\pm q^k$.

Hence the Jones polynomial is uniquely determined if we know the *number* of circles in each state of the cube. To construct the Khovanov complex, we must also know *how these circles bifurcate* when we pass from a state to an adjacent state.

2.2. Atoms. It turns out that all the information needed to calculate the Jones polynomial is contained in the atom corresponding to the virtual diagram.

An *atom* (as introduced by Fomenko [16]) is a pair (M, Γ), where M is a closed two-dimensional manifold and $\Gamma \subset M$ is a four-valent graph (called the *frame*) that divides M into black and white cells in the manner of a chessboard. Atoms are considered up to natural combinatorial equivalence, a diffeomorphism of the corresponding manifolds that preserves the frame and the colouring of cells.

The *vertices* of an atom are vertices of its second component, and the *genus* of an atom is the genus of its first component.

For every virtual diagram L we define an atom $V(L)$ as follows. The vertices of $V(L)$ are in one-to-one correspondence with *classical* crossings of L. The latter are connected by *branches of the diagram*, which may intersect each other (and themselves) at the virtual crossings. There are four branches emanating from each classical crossing. For every branch we connect the corresponding vertices of the atom by an edge. The rule for attaching black (and white) cells is defined by the diagram L. More precisely, let X be a classical crossing of L. Label the four emanating edges x_1, x_2, x_3, x_4 in the clockwise direction in such a way that the branches x_1 and x_3 (resp. x_2 and x_4) locally form an undercrossing (resp. overcrossing). Then we attach black cells to the pairs (x_1, x_2) and (x_3, x_4) of half-edges of the atom.

Figure 2. Two variants of virtualization

Clearly, the atom contains all the information about the number of circles in each state of the diagram. Hence it completely determines the state cube.

The *genus* $g(L)$ of a virtual link diagram L is the genus of the corresponding atom. The genus of a diagram with non-orientable atom may be a half-integer. An important problem is to find the *genus of a virtual link*, the minimal value of the genus of the diagrams representing the link. It is clear from the definition that classical link diagrams of genus zero are connected sums of alternating diagrams.

We define *virtualization* as the local transformation shown in Fig. 2 in the neighbourhood of a classical crossing. Note that virtualization does not change the state cube. Moreover, diagrams with isomorphic atoms can always be obtained from each other by a sequence of detours and virtualizations (see [2], for example).

We shall construct the Khovanov complex starting from a virtual diagram. It will be shown that the homology of this complex is invariant under virtualization (and hence is determined by the atom). This supports the following conjecture in [17]: if a classical diagram L is obtained from a classical diagram L' by a sequence of generalized Reidemeister moves and virtualizations, then the classical links L and L' are isotopic.[2] It follows from what has been said that the Khovanov homology groups of the corresponding links are isomorphic.

In what follows we call this conjecture the *virtualization conjecture*.

2.3. Twisted virtual knots. Twisted virtual knots are generalizations of virtual knots ([14], [18]). They are determined by knots in the orientable thickenings of (possibly non-orientable) two-dimensional surfaces and are considered up to stabilization.

A particular case of the theory of twisted knots is the theory of knots in $\mathbb{R}P^3$. Indeed, given a non-orientable surface S, we have an orientable bundle over S whose fibre is an interval. This bundle is a 3-manifold $S \tilde{\times} I$ with boundary. A good example of such a thickened surface is $\mathbb{R}P^2 \tilde{\times} I$, which is homeomorphic to $\mathbb{R}P^3 \backslash \{*\}$. Thus the Khovanov homology theory for knots in $\mathbb{R}P^3$ is obtained as a particular case.

Let S be a surface and let $\mathcal{S} = S \tilde{\times} I$ be a thickening. Then one can study links in $\mathcal{S}$ by means of their diagrams, that is, their projections on S.

[2]We note that virtually equivalent classical diagrams are equivalent in the usual sense. This follows, for example, from Kuperberg's theorem [5].

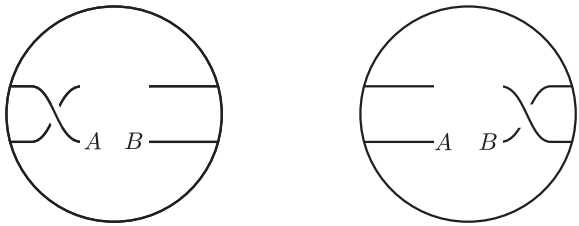


Figure 3. AB is an overcrossing (resp. undercrossing) in the left (resp. right) picture

There are two versions of stabilization/destabilization of knots in thickened surfaces: along orientable cycles and non-orientable cycles. In the second case, an empty 'thickened Möbius band' must be added to/deleted from the surface.

In general position, such a projection is a four-valent graph. To recover the link, we must specify the behaviour of the two branches at every vertex of the graph. The notion of 'being over' is not well defined: travelling along a non-orientable circuit interchanges the upward and downward directions. Hence the notions of an overcrossing or undercrossing have only a relative meaning. To give them an absolute meaning, one can take an affine chart that covers the whole two-dimensional surface except for a one-dimensional part. The upward direction can be canonically defined on the affine chart. When we cross the boundary, this direction may change depending on the orientability of the circuit. For instance, knots in $\mathbb{R}P^3 \backslash * = \mathbb{R}P^2 \mathbin{\widetilde{\times}} I$ may be described by diagrams on $\mathbb{R}P^2$ with all crossings situated inside the disc $D^2 \subset \mathbb{R}P^2$. Passage through the boundary of the disc changes the direction (Fig. 3).

We note that knots in such surfaces are described perfectly by atoms. Indeed, fix an orientation of the thickened surface $S \mathbin{\widetilde{\times}} I$. Then every link diagram in S is the frame of an atom: a four-valent graph with the structure of opposite half-edges at every crossing. The rule for attaching black cells is as follows. Given a vertex X of the atom, take any non-opposite emanating edges a and b. Consider a small vector directed from a to b in the neighbourhood of X. If the orientation of the basis (a, b, c) coincides with that of our 3-manifold, then the angle formed by a and b is declared to be white, and the same for the opposite angle. The remaining two angles are declared to be black. Otherwise the angle between a and b is said to be black (and we attach a black cell). This choice is independent of the pair (a, b) and of the ordering of a, b. Hence we get the following theorem.

Theorem 1. *There is a well-defined map from the set of twisted virtual knots to the set of virtual knots modulo virtualization.*

Knots in such surfaces were considered in [19] and [14]. A Khovanov homology theory for such knots was constructed in [19] by using additional topological information on the surfaces in an essential way.

Theorem 1 and the invariance of the Khovanov homology under virtualization (see Lemma 1 below) imply that the Khovanov homology theory constructed below can be generalized to twisted virtual knots.

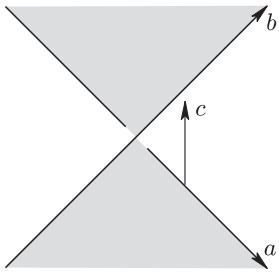


Figure 4. Constructing an atom from a diagram

§ 3. Gauss diagrams and the virtualization conjecture

Virtual knots are generalizations or approximations of classical knots. Many unsolved problems in classical knot theory (say, the problem on knots with trivial Jones polynomial) possess well-known counterexamples in virtual knot theory. As in the case of classical knots, virtual knots may be encoded by Gauss diagrams.

Definition 1. *The Gauss diagram corresponding to a planar virtual knot diagram K consists of an oriented circle (with a marked point) and chords connecting the pre-images of the undercrossing and overcrossing for each crossing. Each chord is endowed with an arrow directed from the pre-image of the overcrossing to that of the undercrossing. Each chord (or arrow) is endowed with a sign, which coincides with the sign of the crossing: it equals* 1 *for*  *and* -1 *for*  .

The pre-images of virtual crossings are not shown in the Gauss diagram. This is consistent with the definition of a detour move: two planar diagrams determine the same Gauss diagram if and only if one of them is obtained from the other by a sequence of detour moves.

If a given Gauss diagram with labelled arrows is realizable by a classical knot diagram, then it uniquely determines a classical knot. If the diagram is non-realizable, then we can consider an *immersion* of the diagram in $\mathbb{R}^2$ instead of an embedding. Of course, we allow only 'good' immersions, without triple points and self-tangencies. Under the immersion, we associate virtual crossings with intersections of the images of edges and we associate classical crossings with the images of crossings (Fig. 5).

Thus we have constructed a virtual knot diagram from the Gauss diagram (possibly non-uniquely).

Theorem 2 [20]. *The isotopy class of a virtual knot is uniquely determined by its Gauss diagram.*

If the Gauss diagram is *realizable* (by a classical knot), then the type of the knot is easily recovered from the signs of the chords of the diagram (without knowing the arrows). This is not the case for virtual knots and immersions: reversing an arrow leads to a virtualization. Hence we can state the virtualization conjecture in terms of Gauss diagrams as follows.

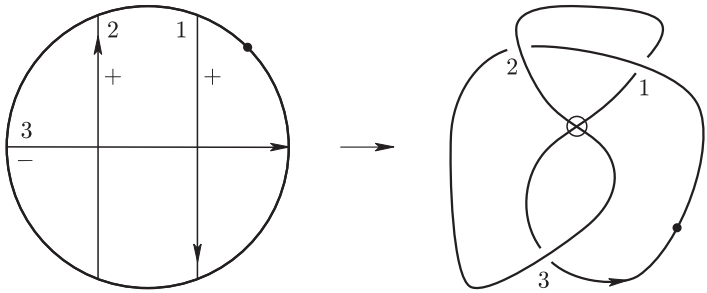


Figure 5. A virtual knot and its Gauss diagram

Consider the set of all Gauss diagrams without arrows and take its quotient modulo the formal relations corresponding to the Reidemeister moves. Is there a one-to-one correspondence between the set of all classical knots and the set of equivalence classes of all realizable Gauss diagrams?

If this is the case, then classical knots may be studied in terms of objects that are simpler than Gauss diagrams: Gauss diagrams without arrows.

We note that Gauss diagrams (realizable and non-realizable) were used in [20] to prove the existence of combinatorial formulae for the Vassiliev invariants of classical knots. These formulae are expressed in terms of Gauss diagrams (realizable and non-realizable). Therefore we may ask whether such explicit combinatorial formulae exist for Gauss diagrams without arrows. One can try to find such formulae at least for Vassiliev invariants that are invariant under virtualization (for example, invariants coming from the Jones polynomials of virtual knots; see [1]).

§ 4. Definition of the Khovanov complex for virtual knots

Our aim is to define a homology theory (with arbitrary coefficients) for virtual knots in such a way that the following properties hold.

1) The homology is invariant under (generalized) Reidemeister moves.

2) The homology of knots with orientable atoms is isomorphic to the Khovanov homology constructed in [10].

3) The tensor product of our complex and $\mathbb{Z}_2$ gives the homology constructed in [9].

A Khovanov homology theory with arbitrary coefficients was constructed in [10] for virtual knots with orientable atoms. The main obstacle in the case of non-orientable atoms stems from bifurcations of the type $1 \to 1$ along edges of the cube.

Remark 1. The role of the coefficient ring in our construction may be played by any commutative ring with identity, for example, by $\mathbb{Z}$. The complex with coefficients in any ring $\mathcal{R}$ is obtained by taking the tensor product of the whole complex and $\mathcal{R}$.

Let V be the two-dimensional graded module over $\mathcal{R}$ with free generators v_+ and v_- of grading $+1$ and -1 respectively.

If there are no bifurcations of type $1 \to 1$, then one can construct the state cube with the usual differentials that correspond to the multiplication m (for bifurcations

of type $2 \to 1$) or to the comultiplication Δ (for bifurcations of type $1 \to 2$) according to [11].

The problem is more difficult if there are $1 \to 1$ bifurcations. (This is characteristic for virtual knot theory and arises from non-orientable atoms.) Indeed, to construct a bigraded theory without changing the grading of the base ring, we need a map from V to V that lowers the grading by 1 (which is then compensated by increasing the grading by 1). Since the basis of V consists of two elements of grading $+1$ and -1, the only map that satisfies these conditions is identically equal to zero. The resulting cube (which will be called the bifurcation cube) must be anti-commutative. It turns out that the zero map of type $1 \to 1$ indeed yields a Khovanov homology theory if the coefficient field is $\mathbb{Z}_2$ (see [9]). In the general case (over $\mathbb{Z}$) one must verify that certain equations are compatible. (We shall see that these equations are related to some two-vertex atoms.) Such a theory has not been constructed without additional assumptions.

In this paper we solve the problem by using two additional constructions: changing the basis in the Frobenius algebra $(1, X)$, which is the homology group of the unlink, as we pass from one crossing to another, and taking the exterior powers of the 'circles' instead of their symmetric powers.

We use the following notation. Given an unordered set of vector spaces, we order them in an arbitrary way: $V_1, \dots, V_n$. Let us define a new space[3] $V_1 \wedge V_2 \wedge \dots \wedge V_n$, which is independent of the ordering. We consider all possible tensor products and identify their elements by the following rule: if $x_i \in V_i$ for $i = 1, \dots, n$, then $x_{\sigma_1} \otimes \dots \otimes x_{\sigma_n} = \operatorname{sign}(\sigma) x_1 \otimes \dots \otimes x_n$. We denote such a tensor product $x_1 \otimes \dots \otimes x_n$ of elements $x_i \in X_i$ by $x_1 \wedge x_2 \wedge \dots \wedge x_n$ and call it an *ordered tensor product*.

Remark 2. To avoid confusion, we note that the notation $X \wedge X$ always assumes that the first vector X and the second vector X are elements of different (although isomorphic) vector spaces. Therefore $X \wedge X$ is not equal to zero, unlike the exterior product of a 1-form and itself.

To make the bifurcation cube anti-commutative, we add two more ingredients.

1. With every circle C in each state we associate a vector space[4] of graded dimension $q + q^{-1}$. Given an orientation o of the circle C in some state, we consider the graded vector space generated by the elements 1 and $X_{C,o}$. Replacing o by the opposite orientation $-o$, we have $X_{C,-o} = -X_{C,o}$.

2. Let s be a state of a virtual link diagram with k circles $C_1, \dots, C_k$. Then we associate the ordered tensor product $V^{\wedge k}$ with this state. We also choose a basis of this product. It is formed by the tensor products $(p^1)_{C_{a_1}} \wedge (p^2)_{C_{a_2}} \wedge \dots \wedge (p^k)_{C_{a_k}}$, where each $(p^i)_{C_{a_i}}$ is an element of the basis of $V_{C_{a_k}}$.

Thus we have defined the space of chains of the complex corresponding to the knot K. We denote this space by $[[K]]$. Its elements correspond to states of the diagram $|K|$. Given a state s of K with circles $C_1, \dots, C_l$, choose elements $\gamma_1, \dots, \gamma_l$ on these circles. This determines a chain in $[[K]]$. We define the height h of this

[3] If the spaces $V = V_1 = \dots = V_n$ coincide, then we also write $V^{\wedge n}$.

[4] From now on, we change the notation v_+ and v_- to 1 and X. (The element v_+ plays the role of the identity under multiplication.) This yields the same homology theory up to a grading shift and normalization. In what follows we ignore these shifts and normalizations, as was done in [21].

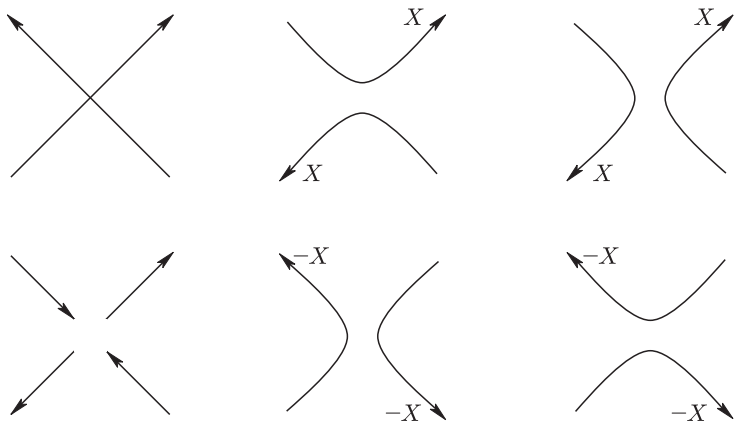


Figure 6. Determining the basis at a crossing

chain to be the number of B-smoothings in the state s. The grading of this chain is defined as $h + \#1 - \#X$, where $\#1$ (resp. $\#X$) is the number of elements 1 (resp. $\pm X$) among $\gamma_1, \ldots, \gamma_l$.

Our next task is to endow this complex with a differential ∂ that increases the height by 1 and preserves the grading.

We introduce partial differentials. Look at classical crossings in such a way that their branches are oriented upwards. Consider a state of an oriented virtual link diagram. Take all circles incident to a given classical crossing. Orient these circles according to the orientation of the upper-right outgoing edge and contrary to the orientation of the lower-left incoming edge (Fig. 6).

Thus the orientation of these circles locally agrees with the orientation of the upper-right outgoing edge (and the lower-right incoming edge) and is opposite to the orientation of the edges to the left of the crossing. We orient half-edges incident to the crossing as shown in the lower-left picture of Fig. 6. An orientation of a circle in a given state is said to be *positive* (at this crossing) if it agrees with the local orientations shown in Fig. 6. Otherwise we say that the circle is *negatively oriented*. This fixes a choice of the generator X on each circle incident to the given crossing. We note that the same generator X for the same circle may have another sign at another crossing.

Using the orientations of the circles at the crossings and the local ordering of the components, we now define the differentials as follows.

The orientation of the circles is well defined unless the edge corresponding to the crossing transforms one circle into one circle. In this case we set the partial differential corresponding to this edge equal to zero.

Now suppose that we have a $1 \to 2$ or $2 \to 1$ bifurcation at the given crossing. When two circles are incident to the crossing from opposite sides, we order them in such a way that the upper (resp. left) circle is the first one. Then the lower (resp. right) circle will be the second. To define the partial differentials, we assume all circles to be ordered in such a way that the circles under consideration are at the beginning of our ordered tensor product. This can always be achieved by

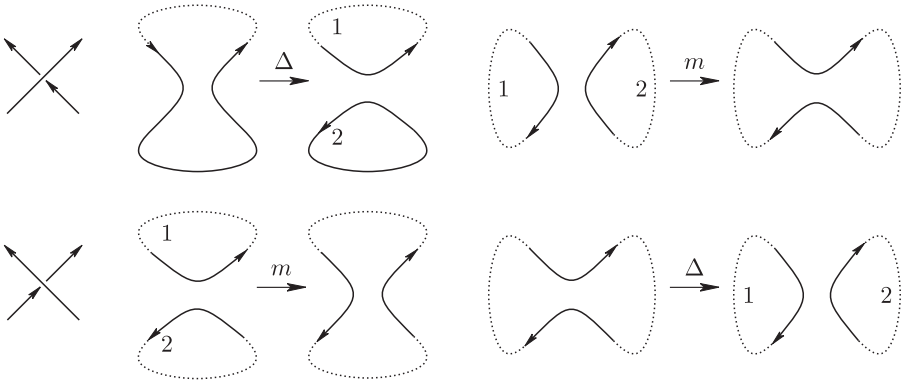


Figure 7. Definition of m and Δ

a permutation, which may possibly change the sign of the result. On the remaining circles, our map acts as the identity map.

Consider an edge (of the bifurcation cube) that changes the number of circles by 1. Since this bifurcation occurs at some crossing, there are two possibilities: either two circles bifurcate to one or one circle bifurcates to two. We have an ordering of the two circles that are incident to the crossing in one of the states. Moreover, all three circles are oriented. This determines a basis in the space corresponding to any of these circles.

Using the sets of generators and the ordering described above, we locally define the maps $\Delta\colon V \to V \wedge V$ and $m\colon V \wedge V \to V$ by setting

$$\Delta(1) = 1_1 \wedge X_2 + X_1 \wedge 1_2, \qquad \Delta(X) = X_1 \wedge X_2,$$
$$m(1_1 \wedge 1_2) = 1, \qquad m(X_1 \wedge 1_2) = m(1_1 \wedge X_2) = X, \qquad m(X_1 \wedge X_2) = 0$$

(see Fig. 7).

If there are circles $C_1, \dots, C_l$ non-incident to the crossing where the bifurcation occurs, and if we are given elements $\gamma_1, \dots, \gamma_k$ on these circles, then the formulae for the partial differentials ∂' are rewritten as

$$\begin{aligned}
\partial'(1 \wedge \gamma_1 \wedge \cdots \wedge \gamma_k) &= \Delta(1) \wedge \gamma_1 \wedge \cdots \wedge \gamma_k \\
&= 1_1 \wedge X_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_k + X_1 \wedge 1_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_k, \\
\partial'(X \wedge \gamma_1 \wedge \cdots \wedge \gamma_k) &= \Delta(X) \wedge \gamma_1 \wedge \cdots \wedge \gamma_k = X_1 \wedge X_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_k, \\
\partial'(1_1 \wedge 1_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_k) &= m(1_1 \wedge 1_2) \wedge \gamma_1 \wedge \cdots \wedge \gamma_k = 1 \wedge \gamma_1 \wedge \cdots \wedge \gamma_k, \\
\partial'(X_1 \wedge 1_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_k) &= \partial'(1_1 \wedge X_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_k) \\
&= m(X_1 \wedge 1_2) \wedge \gamma_1 \wedge \cdots \wedge \gamma_k = X \wedge \gamma_1 \wedge \cdots \wedge \gamma_k, \\
\partial'(X_1 \wedge X_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_k) &= m(X_1 \wedge X_2) \wedge \gamma_1 \wedge \cdots \wedge \gamma_k = 0.
\end{aligned}$$

Example 1. If we wish to comultiply the second factor X_2 in $X_1 \wedge X_2$, then we get $X_1 \wedge X_2 = -X_2 \wedge X_1 \to -X_2 \wedge X_3 \wedge X_1 = -X_1 \wedge X_2 \wedge X_3$, where X_3 belongs to the new (third) component.

§ 5. Statement and proof of the main theorem

Given an oriented virtual link diagram K, we have constructed a set of bigraded groups endowed with a differential ∂ which equals the sum of the partial differentials. We denote this set of groups by $[[K]]$. The differential increases the height and preserves the grading.

In this section we prove the main theorem.

Theorem 3. *The set $[[K]]$ of groups endowed with the differential ∂ is a well-defined bigraded complex, that is, $\partial^2 = 0$. The differential preserves the grading and increases the height by* 1.

Let n_+ (resp. n_-) be the number of crossings of type  (resp. ). Let $\mathcal{C}(K)$ be the complex obtained from $[[K]]$ by the following shifts of height and grading: $\mathcal{C}(K) = [[K]]\{n_+ - 2n_-\}[-n_-]$. Hence the height of every chain is decreased by n_- and the grading is increased by $n_+ - 2n_-$. All differentials are shifted accordingly.

It will follow from the construction below that the homology of $\mathcal{C}(K)$ equals the homology constructed in [10] in the case of virtual knots with orientable atoms. For coefficients in $\mathbb{Z}_2$, the homology of $\mathcal{C}(K)$ obviously coincides with that constructed in [9].

Theorem 4. *The homology groups of the bigraded complex $\mathcal{C}(K)$ are invariants of the link K under generalized Reidemeister moves.*

We first prove Theorem 3 and then give a proof of Theorem 4, which is more technical and follows the standard scheme of [11] modulo the additional verification that the signs (appearing from the ordering and orientations of the circles) of the partial differentials coincide. We shall also prove that the homology of $\mathcal{C}(K)$ equals that constructed in [10] in the case of virtual knots with oriented atoms.

Proof of Theorem 3. We start with two lemmas on some properties of $\mathcal{C}(K)$.

Let K be a virtual diagram. Consider a classical crossing U of K. Let K' be obtained from K by virtualizing U. Then there is a natural one-to-one correspondence between the classical crossings of K and K'. It induces a one-to-one correspondence ϕ between states by choosing the same smoothing (both times A or both times B) at the corresponding crossings. We note that this correspondence preserves the number of circles in each state. This is proved by a local comparison of the smoothings of U in K and K'. (For example, this follows from the fact that the circles in each state are completely described by the atom, and the atom is preserved under virtualization.) This identification determines a map $g\colon [[K]] \to [[K']]$ between the chain spaces. Indeed, for every state s and the corresponding state $\phi(s)$, the diagrams K and K' coincide outside a neighbourhood of U. This yields a one-to-one correspondence between the oriented circles in s and $\phi(s)$ with compatible orientations. Hence the map g is well defined. We use the same notation g for the map between the vector spaces (modules) that correspond to the circles in s and $\phi(s)$.

Every state V of K determines a subspace of the space of chains of $[[K]]$. We denote this subspace by C_V. Let $C_{V'}$ be the corresponding subspace for K'.

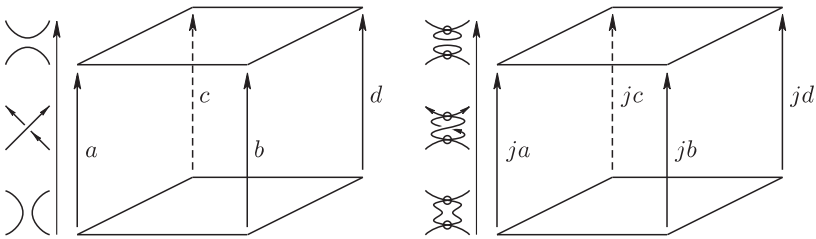


Figure 8. Behaviour of the cube under virtualization, $j = \pm 1$

Lemma 1. *Let K, K' be virtual diagrams obtained from each other by virtualization. Then there is a map $f\colon [[K]] \to [[K']]$ between their chain spaces that preserves the grading, establishes an isomorphism of C_V onto $C_{V'}$ and commutes with the partial differentials. In particular, if $[[K]]$ is a complex, then $[[K']]$ is a complex and their graded homology groups are isomorphic.*

Proof. Suppose that K' is obtained from K by virtualizing a crossing U. The construction of f depends on the type of U (or).

By construction, the partial differentials of $[[K']]$ coincide with the images of those of $[[K]]$ under g, except possibly for those corresponding to the crossing U. The differentials corresponding to U divide the state cube into an 'upper subcube' and a 'lower subcube', as shown in Fig. 8.

The remaining partial differentials can differ only in the signs of the edges corresponding to U. We claim that these differentials either all agree or all differ in sign, as shown in Fig. 8.

Indeed, the bases at all crossings but U agree for K and K'. This yields an identification of chains of the corresponding complexes. Under this isomorphism, we have $g(X_{U,K}) = -X_{U,K'}$ for every circle incident to U. This follows from the fact that if a circle C in a state s of K arrives at U from the upper-right direction, then the corresponding circle $\phi_*(C)$ in state $\phi(s)$ arrives at U from the upper-left direction, which amounts to replacing X by $-X$ in the basis corresponding to this crossing (see Fig 6).

In the case of ordinary tensor products and unordered circles, the passage $X \to -X$ preserves local maps of type m and replaces Δ by $-\Delta$.

We first suppose that the crossing U is positive ( ). All maps of type m for U correspond to a bifurcation of two circles (right and left) into one. After virtualization, the former left circle becomes the right one and vice versa (Fig. 9).

Globally, we see that the differentials of type m change their signs in a fixed basis. For partial differentials of type Δ, one circle bifurcates to two: upper and lower. Since virtualization preserves the relation 'up-down', partial differentials of type Δ are not changed. The first (resp. second) component is shown as a bold line (resp. dashed line) in Fig. 9.

Summarizing, we see that virtualization at a positive crossing changes the signs of all the partial differentials corresponding to this crossing.

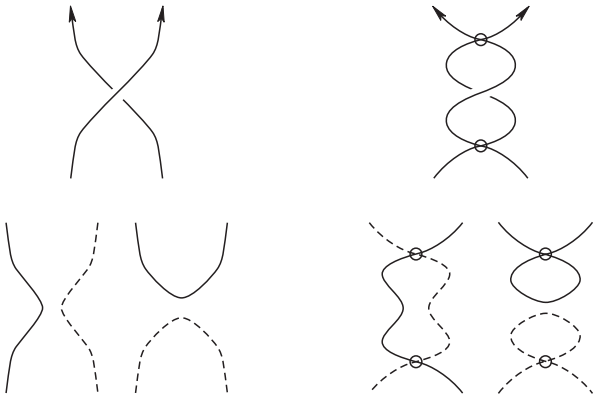


Figure 9. Virtualization

We now divide each of the cubes $[[K]]$, $[[K']]$ into two parts according to the smoothing at U, the lower part and the upper part. We define a map f: $[[K]] \to [[K']]$ which equals g (resp. $-g$) on all elements of the lower (resp. upper) subcube. This map obviously commutes with the partial differentials. If the original cube is anti-commutative, then this map provides a homology isomorphism.

Similar arguments show that virtualization of a negative crossing does not change the cube at all: the minus sign arising from the transposition of the circles (right and left) compensates for the minus sign that appears on the edges corresponding to the partial differentials of type Δ. This proves the lemma.

By Lemma 1, if two diagrams are obtained from each other by virtualization, then their homology groups are isomorphic. Thus, to prove the anti-commutativity of the cube constructed above for a diagram K, it suffices to apply virtualization at some classical crossings and solve the same problem for the resulting diagram K'.

To prove the anti-commutativity of the cube $[[K]]$, we must consider all its two-dimensional faces. Every two-dimensional face corresponds to a way of smoothing $n-2$ classical crossings of K (Fig. 10). The remaining two crossings can be smoothed arbitrarily: the four possible ways of doing this correspond to the four vertices of the face.

The left picture in Fig. 10 shows the bifurcation cube, and the right picture shows a two-dimensional face of it and the corresponding diagram that generates an atom. The atom is determined by the diagram in the way described in § 2.2.

The four previous states have some common circles that are non-incident to either of the two crossings in question. Removing these circles, we get an atom with two vertices. Thus we must check the anti-commutativity for any face that may correspond to an atom with two vertices. We have local orientations of the link at both vertices of such an atom. These orientations determine the partial differentials.

Note that these orientations may disagree when we pass from one vertex to another: every edge of the atom with two vertices is generated by the smoothing of several successive edges, each with its own orientation (Fig. 11).

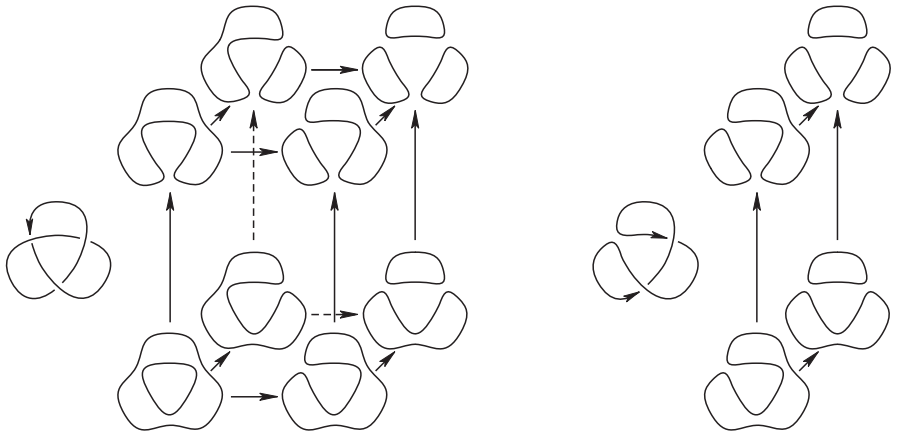


Figure 10. The bifurcation cube (a) and its two-dimensional face (b), which generates an atom

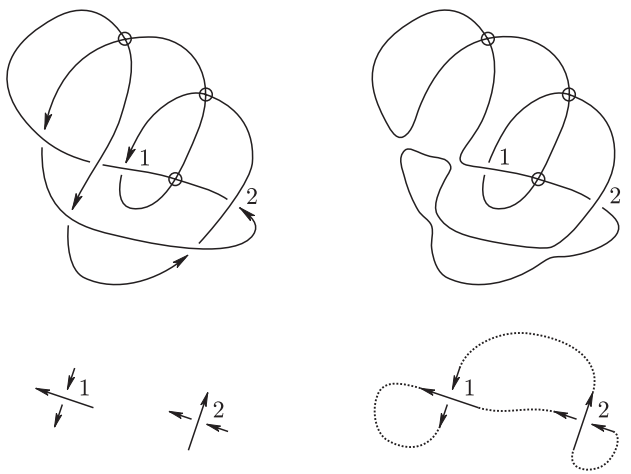


Figure 11. Orientation at crossings of the atom

It turns out that these local orientations may be chosen arbitrarily without loss of anti-commutativity. Namely, fix an atom with two vertices. All possible 'occurrences' of this atom in the cube are determined by local orientations of edges at the vertices. Fix an orientation for one crossing V_1 and take two orientations of the second crossing V_2 which differ from each other by a clockwise rotation through $\frac{\pi}{2}$. We get two squares (two-dimensional cubes), Q_1 and Q_2.

Lemma 2. *If Q_1 is anti-commutative, then so is Q_2. Moreover, there is a map $f\colon [[K]] \to [[K']]$ between the chain spaces that preserves the grading, establishes an isomorphism of C_V onto $C_{V'}$ and commutes with the partial differentials.*

Proof. This mostly repeats the proof of Lemma 1. Here is an outline. Rotating the oriented arrows clockwise at V_2, we change the sign of the generator X on all circles incident to this crossing: as in the proof of Lemma 1, we consider the two complexes and identify their chains by a map g (similar to the map g of Lemma 1) in such a way that the differentials corresponding to the other crossing coincide. Then, arguing as in Lemma 1, we modify the map g to get a map f that commutes with the partial differentials, which will prove Lemma 2.

In the case of the unordered tensor product, the passage from X to $-X$ changes the signs of all partial differentials of type Δ corresponding to the crossing V_2.

For a positive crossing, all differentials of type m at this crossing also change their sign. For a negative crossing, the signs of all multiplications m remain unchanged while the maps Δ change sign again. This is the same situation as in the proof of Lemma 1. The lemma is proved.

Lemma 2 means that to check the anti-commutativity of all two-dimensional faces, it suffices to list all atoms with two vertices and check anti-commutativity for each of them. Namely, we fix a realization of such an atom by immersing its frame in a plane in such a way as to preserve the structure of opposite edges. Such immersions differ by a possible virtualization, which does not influence the anti-commutativity and preserves the homology by Lemma 1. Then we can choose any local orientation of the edges and check the corresponding case.

Note that some atoms with two vertices are disconnected (each of their edges connects some vertex to itself). For such atoms, we get commutative faces in the case of the usual tensor product. However, the ordered tensor products make these faces anti-commutative.

Some (connected) atoms are inessential in the following sense. We have defined each differential of type $1 \to 1$ to be zero. A two-dimensional face of the cube (an atom) may have 0, 2 or 4 such edges. If there are no such edges, then the atom is orientable. If all four edges correspond to differentials of type $1 \to 1$, then the proof is obvious. The proof is also obvious when we have at least one zero map in each of the two composites corresponding to the diagram in question.

The six remaining essential connected atoms with two vertices are shown in Fig. 12. All except the first are orientable.

An accurate calculation (see Fig. 13) shows that both composites are equal to zero in the first case. Indeed, the lower composite is identically equal to zero. Acting on the element X by the upper composite, we get $\pm X \wedge X$ at the first step and zero at the second. Acting on the element 1, we get $1_1 \wedge X_2 + X_1 \wedge 1_2$ at the

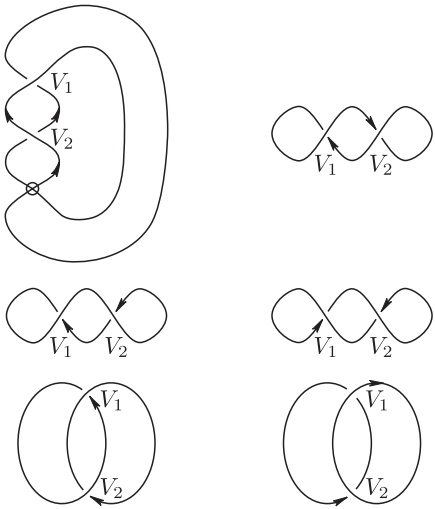


Figure 12. Essential atoms with two vertices

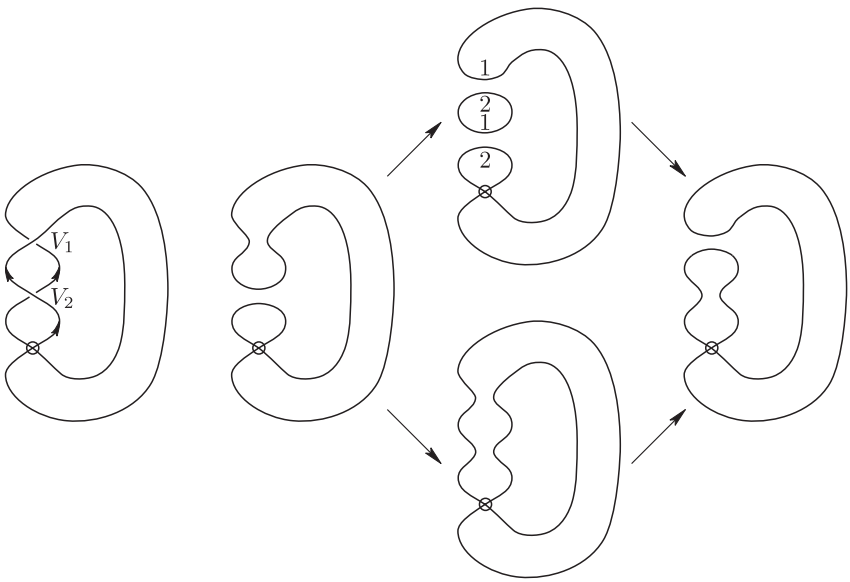


Figure 13. The non-orientable atom

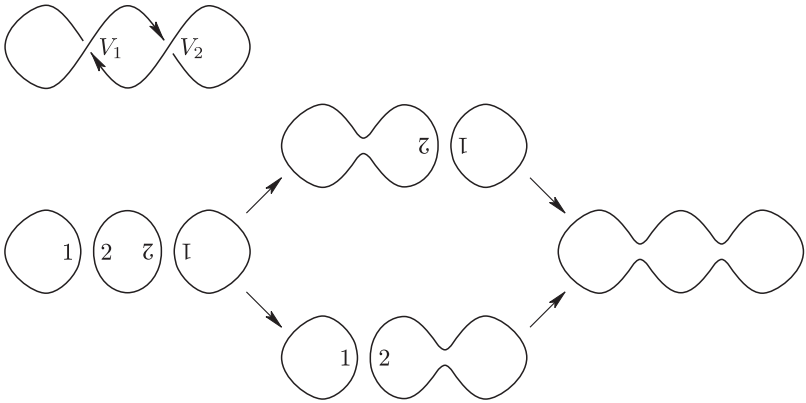


Figure 14. An oriented atom with two vertices

first step. Here the subscripts label the circles (the large circle first and the small one second). When we pass to the second crossing V_2, the first and second circles change their roles: circle 1 becomes the lower one and circle 2 the upper. Moreover, a change of basis occurs for the first circle: X is replaced by $-X$. Thus we get $-X \wedge 1 + 1 \wedge X$, which is transformed into zero by the multiplication m.

Let us check the orientable atoms. We fix their orientations as shown in Fig. 12 and use Fig. 6 to choose the bases $\{1, X\}$ for all circles at the vertices.

Then the anti-commutativity is checked as follows. For an ordinary (unordered) tensor product, we would obtain commutative diagrams. The renumbering of circles may result in the occurrence of minus signs on some edges. We must check that the total resulting sign will be minus for each atom.

For instance, Fig. 14 shows an oriented atom with two vertices. The analogous check for the unordered tensor product is based on the usual associativity formula $m \circ (m \otimes 1) = m \circ (1 \otimes m)$, where the circles are enumerated from left to right. In Fig. 14, one pair 1, 2 of the numbers of the circles is shown upside-down to indicate that the edges are oriented upwards. This also indicates which circle is locally the first (left) and which the second (right).

We must take the global ordering of the components into account. In the case of three components, the multiplication is always applied to the first two, that is, it takes the form $m \wedge \mathrm{id}$. Therefore, applying $m \circ (m \wedge \mathrm{id})$ to $A_1 \wedge A_2 \wedge A_3$, we get

$$m\big(m(A_1, A_2), A_3\big) = -(A_1 \cdot A_2 \cdot A_3).$$

Here, $\cdot$ stands for the usual Khovanov multiplication: $X \cdot X = 0$, $X \cdot 1 = 1 \cdot X = X$, $1 \cdot 1 = 1$. The minus appears because of the second crossing: we have two branches oriented downwards, so the rightmost circle appears locally to be on the left. On the other hand, if we consider the second crossing V_2 first, then we obtain

$$A_1 \wedge A_2 \wedge A_3 = (A_2 \wedge A_3) \wedge A_1 = -(A_3 \wedge A_2) \wedge A_1 \to -(A_2 \cdot A_3) \wedge A_1 = A_1 \wedge (A_2 \cdot A_3).$$

Applying m to the result, we get $A_1 \cdot A_2 \cdot A_3$.

The anti-commutativity of all the other atoms is checked in a similar way. We note that our construction yields the anti-commutative cube directly, unlike Khovanov's original construction which produces an anti-commutative cube from a commutative one by adding some minus signs on the edges. Theorem 3 is proved.

Remark 3. In the following proof of Theorem 4 we ignore all height and grading shifts of the corresponding complexes. The coincidence of the gradings for diagrams that differ by a Reidemeister move is established by repeating the classical case verbatim (see [11], for example).

Proof of Theorem 4. First, we note that the detour move does not change the complex $\mathcal{C}(K)$.

To prove that the homology is invariant under the classical Reidemeister moves, we argue as in the proof of Bar-Natan [11]. The invariance under the first Reidemeister move is obtained by repeating the argument in [11] verbatim. The diagram with an added curl corresponds to the complex

$$[[\,\bigcirc\,]] = ([[\,\bigcirc\,]] \xrightarrow{m} [[\,\bigcirc\,]]\{1\}). \tag{3}$$

The map m is surjective and the subcomplex of $\bigcirc$ on the left with ± 1 on the small circle is mapped by m onto the whole complex $\bigcirc$ on the right. Hence we get an acyclic subcomplex of the original complex. Clearly, the whole complex has the same homology as its quotient by the acyclic part. The homology of this quotient complex is equal to the homology of $[[\,\bigcirc\,]]$.

The other curl is treated in a similar way. Here we use the injectivity of Δ.

The proof of the invariance under the second Reidemeister move also repeats the corresponding argument of Bar-Natan for the classical case. We reproduce it here because it is used in the proof of the invariance under the third Reidemeister move.

Note that we may orient all circles in a locally compatible way such that the variable X does not change its sign as we move from one crossing to another along one circle (Fig. 15).

We continue the argument, taking into account that X does not change sign as we pass from one crossing to another. The original complex $\mathcal{C}$ is given by

$$\begin{array}{ccc} [[\,\bigcirc\,]]\{1\} & \xrightarrow{m} & [[\,\bigcirc\,]]\{2\} \\ {\scriptstyle\Delta}\uparrow & & \uparrow \\ [[\,\bigcirc\,]] & \longrightarrow & [[\,\bigcirc\,]]\{1\} \end{array} \tag{4}$$

It contains a subcomplex

$$\mathcal{C}' = \begin{array}{ccc} [[\,\bigcirc\,]]_1\{1\} & \xrightarrow{m} & [[\,\bigcirc\,]]\{2\} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array} \tag{5}$$

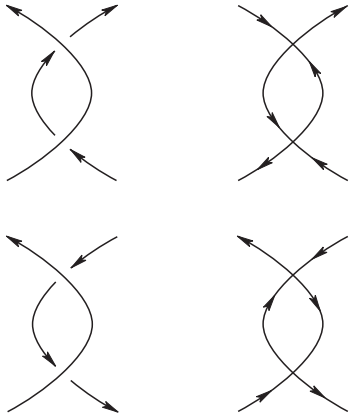
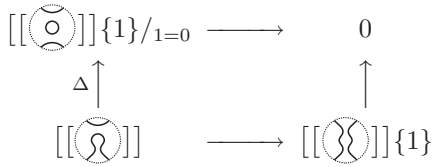


Figure 15. Compatible orientations of edges for the move Ω_2

Here and in what follows, 1 is a label on the small circle. Clearly, the complex $\mathcal{C}'$ is acyclic. Taking the quotient of $\mathcal{C}$ with respect to $\mathcal{C}'$, we get



(6)

The equation $1 = 0$ in the upper-left corner means that we are taking the quotient of the space $\{1, X\}$ (associated with the small circle) modulo the relation $1 = 0$. In other words, we replace the two-dimensional subspace by a one-dimensional one (spanned by X) in the ordered tensor product. Since the arrow Δ in (6) is an isomorphism, we see that (an appropriate normalization of) the complex (6) has the same homology as $[[$  $]]$. This proves the invariance under Ω_2.

We shall use this argument in the proof of invariance of the Khovanov homology under the third Reidemeister move. Returning to the original complex (4), we see that its homology groups are in one-to-one correspondence with the homology groups of the lower-right complex in (6). Moreover, all non-trivial cycles of the original complex have a 'local' height corresponding to the lower-right and upper-left corners of the diagrams. Hence, in the original complex, every element α of the upper-left corner is homologous to exactly one element $-\tau(\alpha)$ of the lower-right corner. The map τ is the composite of Δ^{-1} and the right arrow. To treat the third Reidemeister move, we simplify the subcomplexes that correspond to the second Reidemeister move. This simplification will be done twice. It produces two maps (to be denoted by τ_1 and τ_2) that are similar to τ.

We now consider the third Reidemeister move shown in Fig. 16.

It is well known (see [22], for example) that all variants of the third Reidemeister move are composites of Ω_1, Ω_2 and any single variant of the third Reidemeister move (with a fixed undercrossing-overcrossing structure and fixed orientations of the

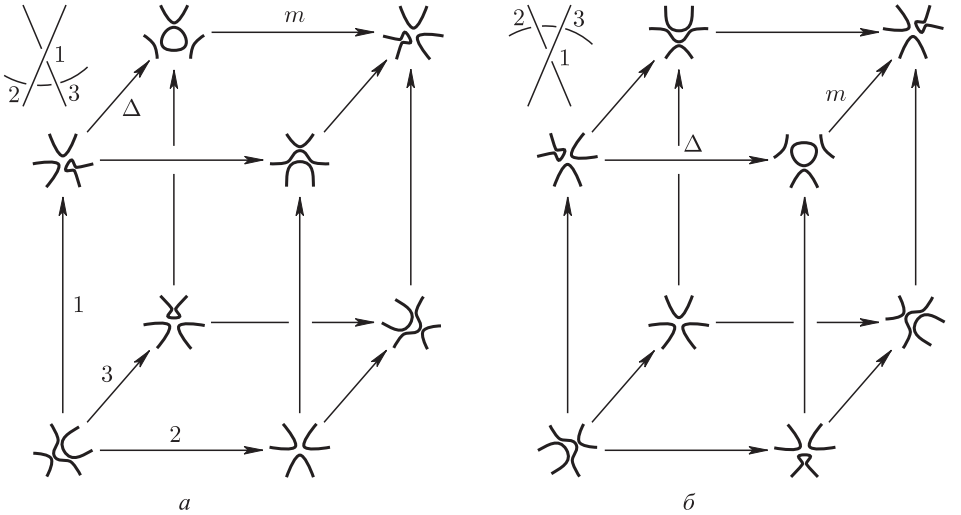


Figure 16. The third Reidemeister move

edges). Hence it suffices to consider only one case: the orientations are shown in Fig. 17 and the crossing structure in Fig. 16.

According to Fig. 17, we have a local orientation rule for all edges incident to any given crossing. These orientations may agree or disagree for two adjacent crossings. For example, the orientation (the upper-right direction) at the third crossing in the left picture is incompatible with the orientations at the first and second crossings. The second crossing in Fig. 16, b is also incompatible with the first and third.

If we virtualize crossings 1, 2 of the diagram in Fig. 16, a and crossing 2 of the diagram in Fig. 16, b, then all the local orientations on both diagrams will be compatible in the sense of the variable X (see Fig. 17).

The positive smoothing of both resulting diagrams at the first crossing (see Fig. 16) yields diagrams that coincide up to virtualization. Negative smoothings yield diagrams that are obtained from each other by (virtualization and) classical Reidemeister moves of the second type. Hence we can transform the complexes (corresponding to both diagrams) so as to guarantee that their lower subcubes coincide and their upper subcubes have the same homology. (We apply Ω_2 in both cases.) It remains to check the compatibility of the partial differentials that are directed upwards: the maps directed upwards either coincide or differ by a sign in both cases.

In the classical case, this was proved in [11]. The only novelty in our situation is the possible occurrence of 'minus' signs because of the ordered tensor product.

In the usual (classical) case, the final complexes (after taking quotients) are described by Fig. 18, which is taken from [11]. The equation $1 = 0$ in the upper-left corner of Fig. 18 means that the space corresponding to this state is replaced by its quotient with respect to the subspace whose small circle contains the element 1. Here τ_1 and τ_2 are not differentials: they are the maps sending each element ζ to an element homologous to $-\zeta$.

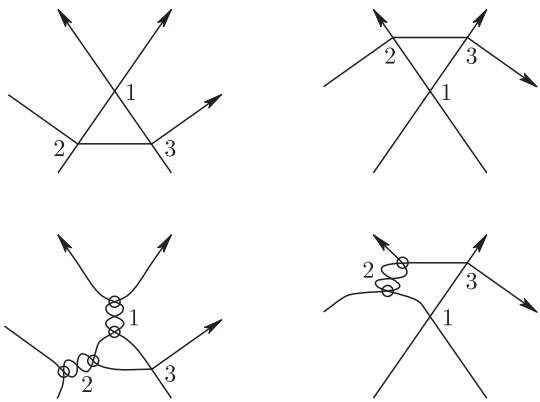


Figure 17. Virtualization of crossings under Ω_3 to make the bases compatible

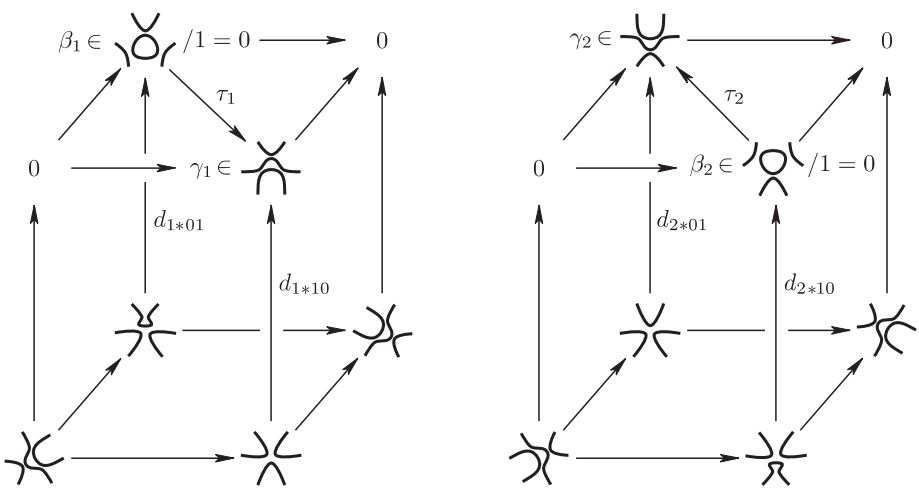


Figure 18. Invariance under Ω_3

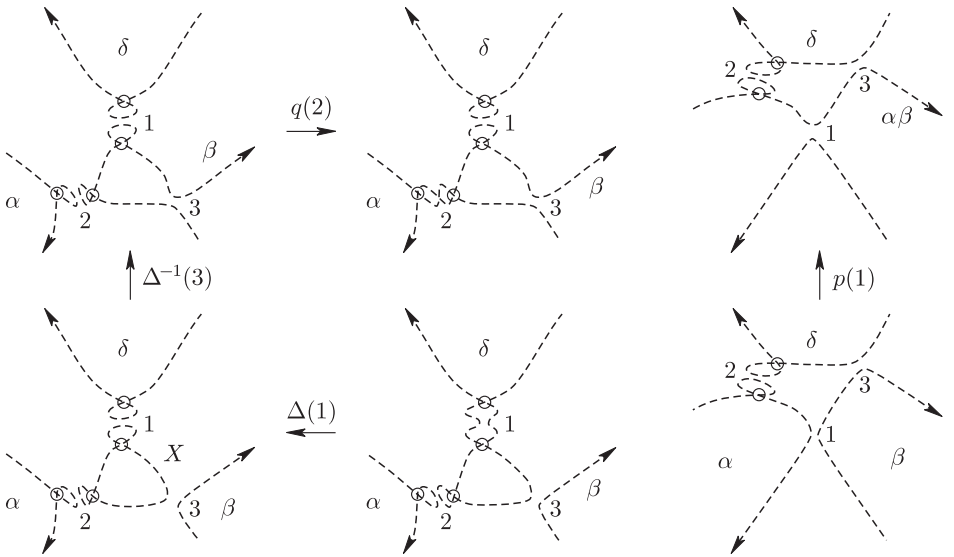


Figure 19. Verification of the invariance under Ω_3

To establish the homology isomorphism, it suffices to prove that

$$\tau_1 \circ d_{1*01} = d_{2*01}, \qquad d_{1*10} = \tau_2 \circ d_{2*10}.$$

In this case, we shall show that the maps directed 'upwards' (on both complexes) differ by sign because both maps τ_i act by $-\mathrm{id}$ in the homology. Then the homotopy equivalence of both complexes corresponding to the third Reidemeister move can be proved similarly to Lemma 1, by means of the natural map that identifies the lower subcubes with sign + and the upper subcubes with sign −.

The case of an ordered tensor product differs from the 'usual' case only in the signs on the edges. Let us show that all signs are compatible. It suffices to prove that $\tau_1 \circ d_{1*01} = d_{2*01}$ (the second case is similar).

Consider Fig. 18. Here d_{1*01} is a partial differential Δ of type $1 \to 2$ and $\tau_1 = v \circ \Delta^{-1}$, where v is a partial differential and Δ^{-1} is the inverse of Δ. (We note that the subspace corresponding to the corner containing β_1 is replaced by its quotient modulo the relation $1 = 0$. In other words, the space corresponding to the small circle C is one-dimensional, with generator X.) The corresponding comultiplication map becomes an isomorphism.

For every map in Fig. 19 we indicate the number (in brackets) of the crossing acted on by this map. The maps p and q are ordinary partial differentials. They are either both multiplications, both comultiplications or both equal to zero. If $p = q = 0$, then the proof is obvious.

Consider the remaining cases. We have three fragments α, β, δ of circles. In the initial state, they can belong to one, two or three different circles. We start with the case when all the fragments α, β, δ belong to different circles. For simplicity, we again denote the elements of V (that is, 1 or $\pm X$) assigned to these circles by the letters α, β, δ. Then both operations p and q are multiplications.

Beginning with $\alpha \wedge \beta \wedge \delta$ in the right picture of Fig. 19, we get

$$p\colon \alpha \wedge \beta \wedge \delta \to (\alpha \cdot \beta) \wedge \delta,$$

where $(\alpha \cdot \beta)$ is the usual multiplication in V. On the left of Fig. 19 we get

$$\alpha \wedge \beta \wedge \delta = \delta \wedge \alpha \wedge \beta \xrightarrow{\Delta} \delta \wedge X \wedge \alpha \wedge \beta.$$

Here we comultiply δ to get two circles at crossing 1. These circles are denoted by δ (upper) and X (lower).

Moreover, $\delta \wedge X \wedge \alpha \wedge \beta = -\beta \wedge X \wedge \alpha \wedge \delta$. The action of Δ^{-1} at crossing 3 yields the union of the two circles to which β and X are assigned. At this crossing, X (resp. β) is assigned to the left (resp. right) circle. Thus we have

$$-\beta \wedge X \wedge \alpha \wedge \delta = X \wedge \beta \wedge \alpha \wedge \delta \xrightarrow{\Delta^{-1}} \beta \wedge \alpha \wedge \delta.$$

The operation q at crossing 2 is a comultiplication, where the circle labelled β is the first (upper) and the circle labelled α is the second (lower). Hence we get $(\alpha \cdot \beta) \wedge \delta$.

Now suppose that α and β belong to one circle (in the initial state) while δ forms a separate circle. We denote the label corresponding to the first (resp. second) circle by A (resp. δ). Applying the map p, we have

$$A \wedge \delta \xrightarrow{\Delta} \sum_i A_{i,1} \wedge A_{i,2} \wedge \delta,$$

where $A_{i,1}$ and $A_{i,2}$ are such that $\sum_i A_{i,1} \otimes A_{i,2}$ is the result of applying the comultiplication to A in the usual sense (see Fig. 20). For brevity, we omit the symbol $\sum_i$ in what follows.

On the left of Fig. 20 we have

$$A \wedge \delta = -\delta \wedge A \to -\delta \wedge X \wedge A.$$

The label δ at crossing 1 corresponds to the upper circle. The other label (always equal to X) corresponds to the lower circle. Applying Δ^{-1} at crossing 3, we get

$$-\delta \wedge X \wedge A = -X \wedge A \wedge \delta \to -A \wedge \delta.$$

Here we had X on the left and A on the right. Finally, the comultiplication q at crossing 2 yields

$$-A \wedge \delta \to -A_{i,1} \wedge A_{i,2} \wedge \delta,$$

where $A_{i,1}$ corresponds to the locally upper component s at crossing 2 while $A_{i,2}$ corresponds to the locally lower component t. However, these two circles meet each other in the opposite way on the right in Fig. 20. More precisely, we have

$$-A_{i,1,s} \wedge A_{i,2,t} \wedge \delta.$$

In the first case (the map p) we had

$$A_{i,1,t} \wedge A_{i,2,s} \wedge \delta = -A_{i,2,s} \wedge A_{i,1,t} \wedge \delta.$$

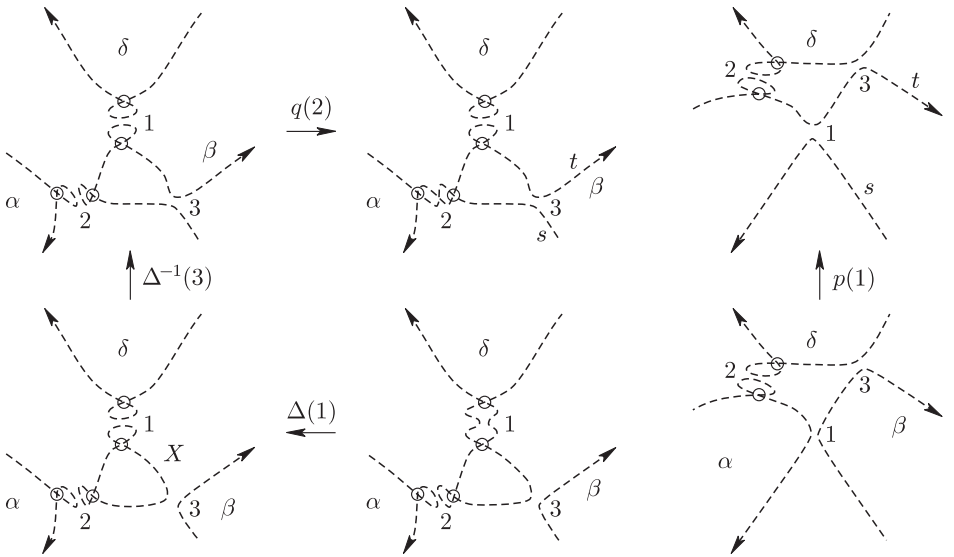


Figure 20. Verification of the invariance under Ω_3

These two results coincide since Δ is cocommutative in the ordinary case. All the remaining cases can be treated in the same way.

Suppose that α and δ belong to one circle (the corresponding element being denoted by α) and β belongs to another circle. Then we have the following maps.

a) We get $\alpha \wedge \beta \to (\alpha \cdot \beta)$ in the simplest case (the map p).

b) On the left in Fig. 20,

$$\alpha \wedge \beta \to \alpha \wedge X \wedge \beta = X \wedge \beta \wedge \alpha \to (\beta \wedge \alpha) \to (\beta \cdot \alpha).$$

Consider the case of multiplication when β and δ form one circle (the corresponding element being denoted by β). Then we get the following maps.

a) On the right in Fig. 20 we have $p\colon \alpha \wedge \beta \to (\alpha \cdot \beta)$.

b) On the left in Fig. 20 we have

$$\alpha \wedge \beta = -\beta \wedge \alpha \to -\beta \wedge X \wedge \alpha = X \wedge \beta \wedge \alpha \to \beta \wedge \alpha \to (\beta \cdot \alpha).$$

Finally, consider the case when there is only one circle (the corresponding element being denoted by A). Then we get two comultiplications.

1) $A \to A_{i,1,t} \wedge A_{i,2,s}$ in the simplest case (the map p).

2) $A \to A \wedge X = -X \wedge A \to -A \to -A_{i,1,s} \wedge A_{i,2,t} = A_{i,2,t} \wedge A_{i,1,s}$.

This proves that $\tau_1 \circ d_{1*01} = d_{2*01}$. The second equation is proved in exactly the same way. Theorem 4 is proved.

Theorem 5. *Let K be a virtual link diagram with an orientable atom. Then the homology groups* $\mathrm{Kh}(K)$ *coincide with the usual Khovanov homology groups constructed in* [10].

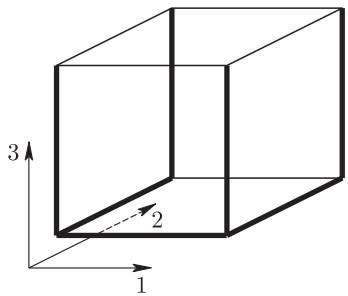


Figure 21. The choice of a spanning tree

To prove this theorem, we denote our new complex and its homology groups by $\mathcal{C}(K)$ and $\mathrm{Kh}(K)$ respectively. The complex and homology constructed in [10] are denoted by $\mathcal{C}'(K)$ and $\mathrm{Kh}'(K)$.

Proof. We can assume that all X are compatible for K, that is, we have $X_P = X_Q$ (and not $X_P = -X_Q$) for every passage from a crossing P to a crossing Q. Indeed, since the atom corresponding to K is orientable, we can globally define an orientation of all edges which is compatible with the orientation of the circles in each state. At every crossing of K, this orientation may agree or disagree with the local orientation of edges determined by Fig. 6. If we virtualize all crossings of K where these orientations disagree, then the homology of $\mathcal{C}(K)$ remains the same by Lemma 1, and the orientations become compatible.

It remains to take care of the signs of the partial differentials and the enumeration of the circles at all crossings.

We construct a homology-preserving map between two cubes. Fix an enumeration of the classical crossings of K. It determines a maximal tree for the cubes $\mathcal{C}(K)$ and $\mathcal{C}'(K)$. This tree consists of all edges of the form $(\alpha_1, \dots, \alpha_k, *, 0, \dots, 0)$, where $\alpha_j \in \{0, 1\}$. In other words, the tree contains an edge in the direction x_j if all the coordinates $x_{j+1}, \dots, x_n$ vanish (Fig. 21).

With every state s of the complex $\mathcal{C}(K)$ we associate the ordered tensor power $V^{\wedge k}$, where k is the number of circles. With the corresponding state of $\mathcal{C}'(K)$ we associate the space $V^{\otimes k}$. Enumerate the circles in state A in some way. Then the ordering determines a map between the space corresponding to the A-state s in $\mathcal{C}(K)$ and the space corresponding to some state $g(s)$ of $\mathcal{C}'(K)$. Successively renumbering the circles at all vertices of the tree, we can assume that the identifications of complexes in the corresponding states of $\mathcal{C}(K)$ and $\mathcal{C}'(K)$ are compatible with the partial differentials acting along the edges of the spanning tree. Thus we have constructed a map g between the whole chain space of $\mathcal{C}(K)$ and the chain space of $\mathcal{C}'(K)$.

This map commutes with all the partial differentials for the following reason. Let ∂', ∂'' be the partial differentials corresponding to the same edge of the complexes $\mathcal{C}$ and $\mathcal{C}'$. Then we have

$$g \circ \partial' = \pm \partial'' \circ g.$$

If the compatibility holds for three edges of some two-dimensional face of the cube, then it also holds for the fourth edge since both complexes are anti-commutative. To complete the proof, we note that all the edges of the cube can be exhausted if we start from the maximal tree and successively add the missing edges of the two-dimensional faces (add the fourth edge provided that we have three).

§ 6. Generalizations

6.1. The maximal tree for Khovanov homology. Some constructions and results of the Khovanov homology theory can be extended directly to the virtual link homology theory constructed in this paper. These results include the maximal tree theorem suggested independently by Wehrli [23] and Kofman and Champanerkar [24], as well as the results on the minimality of diagrams described in [25].

More precisely, it was shown in [23] that the Khovanov homology is isomorphic to the homology of a certain complex. Let $\mathcal{K}_1(L)$ be the set of all states of the diagram L in which the number of circles is equal to 1.

Lemma 3. *The bigrading of non-zero Khovanov homology* $\mathrm{Kh}(L)$ *can only take values of the form* $(C_1 + \beta - w, C_2 + \beta - 2w \pm 1)$, *where* w *belongs to some finite set of integers,* β *belongs to the set of values* $\beta(s)$ *over all states* $s \in \mathcal{K}_1(L)$ *and* C_1, C_2 *are constants.*

Given a Laurent polynomial in one variable, we denote by span the difference between the maximal and minimal exponents of its non-zero monomials.

A diagram L with n crossings is said to be 1-*complete* if $\mathrm{span}\langle L\rangle = 4n - 4g(L)$, where g is the genus of the atom corresponding to L. An oriented diagram is said to be 1-*complete* if the corresponding non-oriented diagram is 1-complete.

A proof of Lemma 3 for classical knots was given in [23]. It extends verbatim to the case of virtual knots. An important particular case of this lemma is an assertion on the thickness of the Khovanov homology.

Consider the Khovanov homology of an oriented link L over some non-graded ring R. Let $t_{\max}$ and $t_{\min}$ be the maximal and minimal values of $2x - y$ over all pairs (x, y) such that the Khovanov homology group of L with bigrading (x, y) is non-trivial. The number $(t_{\max} - t_{\min})/2 + 1$ is called the *thickness* of the Khovanov complex and is denoted by $T_R(L)$ (see [26], [27]). This number is an integer for virtual links with orientable atoms and may be a half-integer for links with non-orientable atoms.

The *thickness* $T(L)$ of an oriented link diagram L is defined as maximum of the thicknesses over all coefficient rings.

Lemma 3 and the definition of an atom yield the following lemma.

Lemma 4. *We have* $T(L) \leqslant g(L) + 2$ *for every connected*[5] *virtual link diagram* L.

A virtual link L is said to be 2-*complete* if $g(L) + 2$ equals $T(L)$.

Lemma 5. *We have* $\mathrm{span}\langle L\rangle \leqslant 4n - 4g(L)$ *for every connected virtual diagram* L *with* n *crossings.*

[5] A diagram is said to be *disconnected* if the corresponding graph becomes disconnected after a sequence of detour moves.

Proof. See [2], for example.

The following theorem is a corollary of Lemmas 5 and 4.

Theorem 6. *Let L be a connected virtual link diagram. Suppose that L is 1-complete and 2-complete. Then it has the minimal number of classical crossings in the class of connected diagrams.*

6.2. Link homology and Frobenius extensions. Some natural extensions of the Khovanov theory of classical knots were suggested by Khovanov himself. Some of them can be extended to virtual knots. The main results of this subsection were obtained in [10] in the case of orientable atoms.

Let $\mathcal{R}$, $\mathcal{A}$ be commutative rings and let $\iota\colon \mathcal{R} \to \mathcal{A}$ be a ring embedding with $\iota(1) = 1$. Consider the restriction functor that sends $\mathcal{A}$-modules to $\mathcal{R}$-modules. It has right and left adjoint functors: the induction functor $\mathrm{Ind}(M) = \mathcal{A} \bigotimes_{\mathcal{R}} M$ and the coinduction functor $\mathrm{CoInd}(M) = \mathrm{Hom}_{\mathcal{R}}(\mathcal{A}, M)$. We say that ι is a Frobenius embedding if the induction and coinduction functors are isomorphic. Equivalently, the embedding ι is Frobenius if the restriction functor has a two-sided dual functor. Then the ring $\mathcal{A}$ is called the *Frobenius extension* of $\mathcal{R}$ by means of ι.

Khovanov [21] asked which pairs $\mathcal{A}$, $\mathcal{R}$ of rings possess the following property: if we take $\mathcal{R}$ for the basic coefficient ring and $\mathcal{A}$ (a Frobenius algebra over $\mathcal{R}$) for the main building block (the homology group of the unknot), then we can construct an invariant homology theory for classical links 'in the same way'. The expression 'in the same way' means that we consider the state cube, associate with its vertices the tensor powers of $\mathcal{A}$ (over $\mathcal{R}$) with exponent equal to the number of circles in the given state, define the partial differentials by means of the multiplication and comultiplication maps, attach signs to the edges of the cube and normalize the whole construction by shifts.

Khovanov proved that the invariance under the first Reidemeister move forces $\mathcal{A}$ to be a two-dimensional $\mathcal{R}$-module. He gave necessary and sufficient conditions for the existence of such homology theories. It is shown in [21] that every solution is obtained using certain operations (basis change, twisting and duality) from the *universal* solution $(\mathcal{R}, \mathcal{A})$, which is defined by

$$\mathcal{R} = \mathbb{Z}[h,t], \qquad \mathcal{A} = \mathcal{R}[X]/(X^2 - hX - t),$$
$$\deg X = 2, \qquad \deg h = 2, \qquad \deg t = 4,$$
$$\Delta(1) = 1 \otimes X + X \otimes 1 - h \cdot 1 \otimes 1,$$
$$\Delta(X) = X \otimes X + t \cdot 1 \otimes 1.$$

Thus the multiplication in the algebra $\mathcal{A}$ preserves the grading and comultiplication increases it by two. We shall not describe the normalizations for the grading.

We call this construction the *universal* $(\mathcal{R}, \mathcal{A})$*-construction*. The corresponding homology of a classical oriented link L is denoted by $\mathrm{Kh}_U(L)$.

Khovanov's main idea [21] is as follows. First he studies the structure of Frobenius extensions from the viewpoint of the invariance of the resulting homology theory under the first Reidemeister move Ω_1. It follows that $\mathcal{A}$ is a two-dimensional $\mathcal{R}$-module. He then considers the universal topological construction of Bar-Natan [13] and constructs a functor from Bar-Natan's topological category to

the category of Frobenius extensions of rank two. This functor is neither surjective nor injective. However, it preserves all the properties necessary for the invariance of the homology under Reidemeister moves.

Thus Khovanov shows that any link homology theory based on rank-two Frobenius extensions can be reduced (without loss of information) to the universal theory by certain algebraic operations.

Without going into the details of the constructions of Khovanov and Bar-Natan, we take the universal $(\mathcal{R}, \mathcal{A})$-construction and the structural laws of Khovanov's theory as a basis on which to construct a homology theory for virtual links.

We note that Khovanov also studied the question of whether his new theory is functorial, that is, has 'good behaviour' under cobordisms (projective functoriality). To do this, he introduced the unity and co-unity maps (in addition to the operations of multiplication and comultiplication) and considered transformations of them. We do not consider this aspect here. Neither do we describe the meaning of the algebraic operations under Frobenius extensions ($\mathcal{A}$ over $\mathcal{R}$): twisting, basis change and conjugation. We only note that each of these operations leads to a link homology which is recovered from the link homology calculated in the original theory.

The following theorem is proved by direct verification.

Theorem 7. *The restriction of Khovanov's universal theory to the case $h = 0$ (no restrictions on t) can be extended to virtual links by the method suggested in the present paper.*

Here is the main idea of the proof of Theorem 7. The involution $I\colon 1 \to 1$, $X \to -X$, which results from changing the orientation of a circle, preserves the map m and changes the sign of Δ. This holds only for $h = 0$ without restrictions on t. An analogous theory for coefficients in $\mathbb{Z}_2$ was constructed in [10] without any restrictions on h and t. This theory extends to the case when $h = 0$ (and all differentials of type $1 \to 1$ are assumed to be zero). As a particular case, we get an analogue of Lee's theory [28].

§ 7. Discussion and open questions

All the proofs in [7], [8] are *local*. It follows from these papers that all the theories constructed there extend to the case of virtual knots (by means of matrix factorizations). However, there is still no explicit description of these theories without matrix factorizations and their tensor products, even for classical knots. Therefore the following remarks are important.

1. The Khovanov–Rozansky homology is invariant under virtualization in the general case. This is further evidence for the virtualization conjecture. This follows implicitly from [29].

2. The projective functoriality (that is, functoriality up to a sign) of various Khovanov and Khovanov–Rozansky homology theories is proved in [7], [12], [13]. In future publications, we shall consider the problem of projective functoriality for the theory constructed in this paper.

3. An important problem is to construct a 'topological Khovanov theory' for non-oriented cobordisms (as in [13]) in such a way that the construction in this paper is a realization of it. In other words, one must construct a complex of *non-oriented*

cobordisms of ordered oriented circles. This theory was partially constructed by Turaev and Turner [30], for example, in the case of the field $\mathbb{Z}_2$.

An exact proof of functoriality and the construction of a topological cobordism theory will be given in a future publication.

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