Complex manifolds

Def X is complex manifold if

X is differentiable manifold

given by an equiv class of atlases

with charts (U_i, f_i) of form

$$f_i : U_i \xrightarrow{\sim} f_i(U_i) \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$$

and with transition functions

$$g_{ij} = f_i \circ f_j^{-1} : f_j(U_{ij}) \to f_i(U_{ij})$$
$$U_{ij} = U_i \cap U_j$$

that are holomorphic.

On each $\mathbb{C}^n$, have usual coordinates $z_a = x_a + iy_a$

Multiplication by i gives an automorphism

$$I : T\mathbb{C}^n \to T\mathbb{C}^n, \quad \partial_{x_a} \mapsto \partial_{y_a}, \quad \partial_{y_a} \mapsto -\partial_{x_a}$$

of the tangent space $T\mathbb{C}^n$ satisfying $I^2 = -1$.

This induces a vector bundle automorphism

$$I : TX \to TX$$

of the tangent bundle TX satisfying $I^2 = -1$.

Rem Alternatively, we may start with a $2n$-manifold M with such an I. To get a complex n-fold, it must satisfy a further condition of "integrability"

Kähler manifolds

First we have:

Def X is a hermitian manifold if it has a metric g compatible with I,

namely on each tangent space have

$$g(-,-) = g(I-, I-)$$

Then we define a <u>fundamental form</u>

$$\omega(-,-) = g(I-,-)$$

as a section of $\wedge^2 T^\vee X$

<u>Def</u> X is a <u>Kähler manifold</u> if

ω is closed, namely $d\omega = 0$.

<u>Ref</u> For more details about calculus on

manifold, such as exterior derivative d,

see for instance [Huybrechts, Complex Geometry]

Note the metric g is determined by I and ω, via

$$g(-,-) = \omega(-, I-).$$

<u>Ex</u> $\mathbb{P}^n$ has canonical such metric g,

called Fubini–Study (pronounced "Stoody") metric.

<u>Def</u> Say submanifold $Y \subset X$ is <u>complex submanifold</u> if

I maps $TY \to TY$ in TX.

<u>Note</u> Kähler structure restricts naturally to

complex submanifolds.

<u>Ex</u> Any projective manifold is Kähler.

Using this, we can construct Calabi–Yau 3-folds

in the sense of previous lecture.

Ex Quintic 3-fold $Y \subset X = \mathbb{P}^4$ is CY3.

By definition Y is zeroes of generic section of line bundle $\mathcal{O}(5)$ on $\mathbb{P}^1$. Hence normal bundle $N_{Y|X} \cong \mathcal{O}(5)|_Y$. But $\omega_X = \det T^\vee X \cong \mathcal{O}(-5)$, so adjunction formula gives $\omega_Y \cong \mathcal{O}$.

Kähler manifolds are complex manifold by definition, but also have structure of symplectic manifold.

This point of view is key to our understanding of mirror symmetry.

Symplectic vector spaces

V: finite-dimensional $\mathbb{R}$-vector space

ω: 2-form on V, namely $\omega \in \Lambda^2 V^\vee$

Note ω here not to be confused with bundle ω_X above.

We obtain linear map

$$\phi : V \longrightarrow V^\vee$$
$$v \longmapsto \omega(v, -)$$

Def V with form ω is symplectic if ϕ is isomorphism.

Fact symplectic iff $\dim V = 2n$ and V has basis

$e_1 \dots e_n, f_1 \dots f_n$ where $\omega(e_i, e_j) = 0$

$$\omega(f_i, f_j) = 0$$
$$\omega(e_i, f_j) = \delta_{ij}$$

Fact symplectic iff $\dim V = 2n$ and $\omega^n \neq 0$.

Proof of "only if". Take coordinates x_i, y_i dual to e_i, f_i.

Then $\omega = \sum_i dx_i \wedge dy_i$, $\omega^n = n! \bigwedge_i dx_i \wedge dy_i \neq 0$.

Lagrangian vector subspaces

For vector subspace $U \subseteq V$ write symplectic orthogonal

$$U^{\omega} = U^{\perp} = \{ v \in V \mid \omega(-, v) = 0 \}$$
$$= \phi^{-1} U^{0}$$

for annihilator U^0. From $\dim U + \dim U^0 = \dim V$ we get

$$\boxed{\dim U + \dim U^{\omega} = \dim V.}$$

Say U is <u>Lagrangian</u> if $U = U^{\omega}$
<u>isotropic</u> if $U \subseteq U^{\omega}$ $\Big\}$ imply $\omega|_U = 0$
<u>coisotropic</u> if $U^{\omega} \subseteq U$

These imply $\dim U = n$
$\leq n$
$\geq n$ respectively, for $\dim V = 2n$