Towards homological mirror symmetry (HMS)

So far, we have discussed mirror symmetry (MS) as a relation between

- numbers, $h^{p,q}$, or
- vector spaces, $H^{p,q}$

associated to mirror partners X and X'. For this, the same type of object (we could say "type of invariant") is used for X and X'.

A further relation is given by considering categories associated to X and X'. A different construction is used on each "side of the mirror", as follows.

Recall the mirror map $\mathcal{M}_K(X) \cong \mathcal{M}_C(X')$ suggested that mirror symmetry relates symplectic geometry on X to complex geometry on X'

Recall Calabi-Yau 3-fold X has

$\boxed{\text{complex structure } J, \text{ metric } g, \text{ Kähler form } \omega.}$

and in physics terms we have

- A-model: determined by symplectic manifold (X, ω) (and B-field)
- B-model: determined by complex manifold (X, J)

We attempt to construct categories $\mathcal{C}_A(X,\omega)$ and $\mathcal{C}_B(X,J)$ which mathematically describe (some aspects of) the A- and B-model.

<u>Rem</u> This idea was promoted by Kontsevitch in the following (difficult) article from '94:

[K] Kontsevitch, Homological algebra of mirror symmetry

<u>Def</u> A category $\mathcal{C}$ has

- objects $\mathrm{Ob}\,\mathcal{C}$
- morphisms $\mathrm{Mor}\,\mathcal{C}$

We require each $f \in \mathrm{Mor}\,\mathcal{C}$ to have a source and target, written $f: x \to y$ or $x \xrightarrow{f} y$

We also write $f \in \mathrm{Mor}(x, y)$.

We have identities $x \xrightarrow{id} x$ and composition of $x \xrightarrow{f} y \xrightarrow{g} z$ to give $x \xrightarrow{h} z$

<u>Ex</u> A group G gives a category with

$$\mathrm{Ob}\,\mathcal{C} = \{*\}, \quad \mathrm{Mor}\,\mathcal{C} = G$$

<u>Ex</u> We may define a category $\mathrm{Vec}_{\mathbb{C}}$ with

$$\mathrm{Ob}\,\mathcal{C} = \{\text{vector spaces} / \mathbb{C}\}$$

$$\mathrm{Mor}\,\mathcal{C} = \{\text{morphisms between them}\}$$

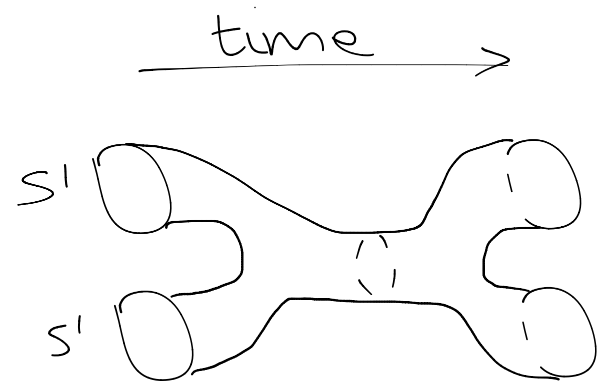
<u>Ex</u> As a generalization, for a complex manifold X we may define $\mathrm{Bun}(X)$ with

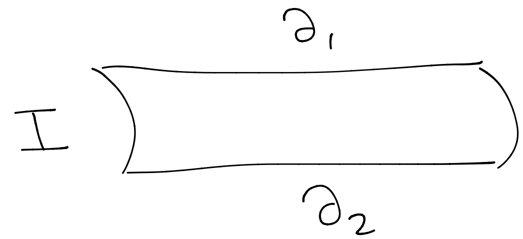
$$\mathrm{Ob}\,\mathrm{Bun}(X) = \{\text{complex vector bundles over } X\}$$

<u>Rem</u> For an algebraic variety X (for instance a toric variety), we may similarly define $\mathrm{Bun}(X)$ by restricting to <u>algebraic</u> vector bundles

Strings and branes

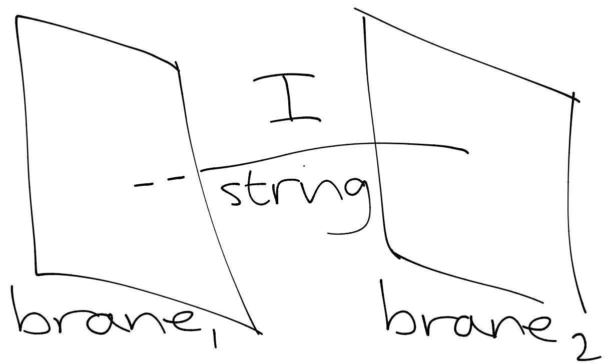
In string theory, as well as closed strings ($\cong S^1$), we study open strings ($\cong I = [0,1]$).





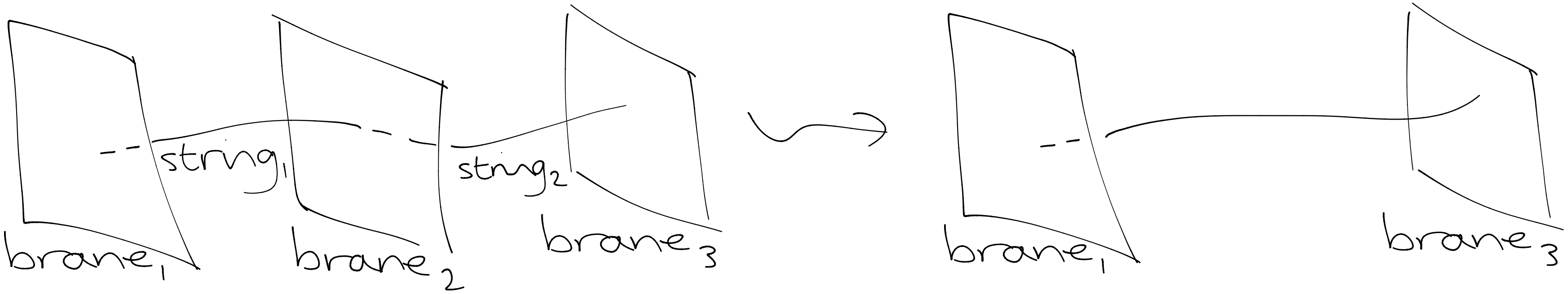
The equations governing these have certain boundary conditions governing, in particular, the positions of the ends $0, 1 \in I$.

at fixed time t:



As string theory developed, these were recognized as interesting objects in themselves, and named "branes".

It is expected that the branes occuring for a given spacetime M are $\mathrm{Ob}\,\mathcal{C}$ for some sort of category $\mathcal{C}$, and the strings with ends on them are $\mathrm{Mor}\,\mathcal{C}$. We imagine "composition" of strings as follows.



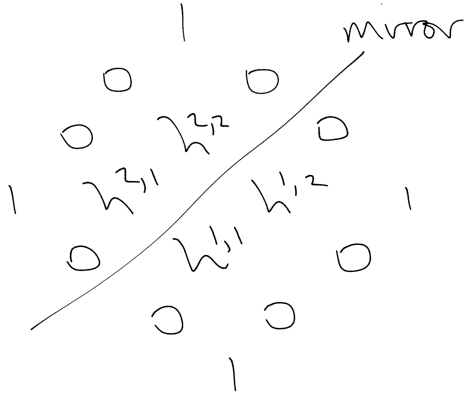
But what is the category? As a first approximation, we can say

(A) $\quad \mathrm{Ob}\,\mathcal{C}_A(X,\omega) \supset \mathrm{Lag}(X,\omega)$

(B) $\quad \mathrm{Ob}\,\mathcal{C}_B(X,J) \supset \mathrm{Ob}\,\mathrm{Bun}(X,J)$

Here $Lag(X,\omega)$ is the set of Lagrangians in the symplectic manifold (X,ω).

<u>Rem</u> To give an idea of why this might work, note that (A) a Lagrangian L determines a class $[L]$ in $H^3(X,\mathbb{C}) = \bigoplus_{p+q=3} H^{p,q}(X)$, whereas (B) a bundle B determines a "Chern character" $ch(B)$ in $\bigoplus_p H^{p,p}(X)$ and these groups are exchanged by mirror symmetry.



However, this rough analysis also suggests we should include further objects, to relate to the rest of the Hodge diamond.