Property of the Seiberg-Witten invariants

- X : smooth, Riemannian metric g
- $\mathfrak{s}$: spin$^{\mathbb{C}}$ structure
- The Seiberg-Witten map

$$\widetilde{SW} : \{\text{spin}^{\mathbb{C}} \text{ connections}\} \oplus \Gamma(S^+) \to \Omega^2_+(X; i\mathbb{R}) \oplus \Omega^0(X; i\mathbb{R})/_{i\mathbb{R}} \oplus \Gamma(S^-)$$
$$(A, \phi) \longmapsto (F^+_{A^t} - \rho^{-1}(\phi\phi^*)_0,\ d^*(A - A_0),\ \not{D}_A \phi)$$

The Seiberg Witten moduli space $\mathcal{M}_{SW} = \widetilde{SW}^{-1}(\eta, 0, 0)/\mathcal{G}_h$

$\eta \in \Omega^2_+(X; i\mathbb{R})$ perturbation 2-form

$\mathcal{G}_h = \{\text{harmonic maps } X \to U(1)\} = H^1(X;\mathbb{Z}) \times S^1$

We have sketched a proof of the following (when $b_1(X) = 0$)

- Theorem: When $b^+(X) > 0$, $\exists$ a generic η s.t.
  $\mathcal{M}_{SW}$ is a compact orientable manifold of dimension
$$d(\mathfrak{s}) := \frac{c_1(\mathfrak{s})^2 - (2\chi(X) + 3\sigma(X))}{4}$$

It's orientation is specified by
an orientation of $H^1(X;\mathbb{R}) \oplus H^2_+(X;\mathbb{R})$ ← homology orientation

When $b^2_+(X) > 1$, $d(\mathfrak{s}) = 0$, we define $SW(X, \mathfrak{s}) = \# \mathcal{M}_{SW}(X, \mathfrak{s})$

when $d(\mathfrak{s}) > 0$, we define $SW(X, \mathfrak{s}) = c_1(P)^{\frac{d(\mathfrak{s})}{2}} \cdot [\mathcal{M}_{SW}(X, \mathfrak{s})]$.

where $S^1 \hookrightarrow P \to \mathcal{M}_{SW}$

$$P = \widetilde{SW}^{-1}(\eta, 0, 0)/\mathcal{G}_{h,b} := \{\text{harmonic } X \to S^1 \text{ s.t. } b \mapsto 1\}$$

base point

Proposition: When $b^+(X)>1$, the Seiberg-Witten invariant does not depend on the choice of (g, η) ← metric ← perturbation.

Proof: Let (g,η) be an admissible pair. I.e, a pair that satisfies

$$P_{H_g^+}(\eta + 2\pi i c_1(\mathfrak{S})) \neq 0$$

$P_{H_g^+} : i\Omega^2(X) \longrightarrow iH_g^+$ orthogonal projection to the space of self-dual 2-form.

Since $\dim H_g^+ = b_2^+(X) > 1$, any two admissible pair can be connected by a path $\{(g_t, \eta_t)\}_{t\in[0,1]}$ of admissible pairs.

Consider $\widetilde{\mathcal{M}}_{SW} = \bigcup_{0\le t\le 1} \mathcal{M}_{SW}(g_t,\eta_t)$. This is a cobordism from $\mathcal{M}_{SW}(g_0,\eta_0)$ to $\mathcal{M}_{SW}(g_1,\eta_1)$

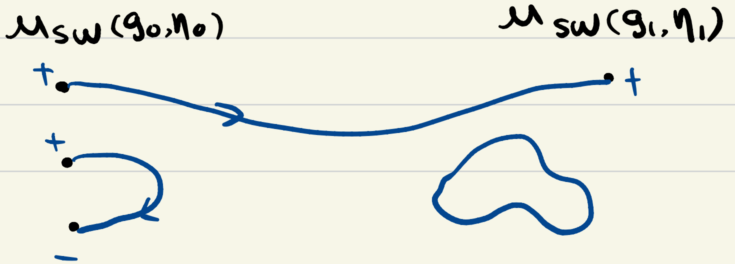
so $[\mathcal{M}_{SW}(g_0,\eta_0)] = [\mathcal{M}_{SW}(g_1,\eta_1)] \in H_*(\widetilde{\mathcal{M}}_{SW};\mathbb{Z})$.

By putting $S^1 \hookrightarrow P_t \to \mathcal{M}_{SW}(g_t,\eta_t)$ together, we can form

$$S^1 \hookrightarrow \tilde{P} \to \mathcal{M}_{\widetilde{SW}}.$$

Then
$$\begin{aligned} SW(g_0,\eta_0) &= c_1(P_0)^{\frac{d(\mathfrak{S})}{2}}[\mathcal{M}_{SW}(g_0,\eta_0)] \\ &= c_1(\tilde{P})^{\frac{d(\mathfrak{S})}{2}}[\mathcal{M}_{SW}(g_0,\eta_0)] \\ &= c_1(\tilde{P})^{\frac{d(\mathfrak{S})}{2}}[\mathcal{M}_{SW}(g_1,\eta_1)] \\ &= SW(g_1,\eta_1). \end{aligned}$$ $\square$

Picture when $d(\mathfrak{S})=0$



When $b^+(X)=1$, the space $\{(g,\eta) \mid P_{H_g^+}(\eta+2\pi i c_1(\mathfrak{s}))\neq 0\}$

has two components (each called a "chamber")

$SW(g,\eta)$ can take two values.

If we orient $H^2_+(X;\mathbb{R})\cong\mathbb{R}$. Then we can talk about

$SW^+(X;\mathfrak{s}) = SW(g,\eta)$ for $P_{H_g^+}(\eta+2\pi i c_1(\mathfrak{s}))>0$

$SW^-(X;\mathfrak{s}) = SW(g,\eta)$ ---- —— —— -- —— <0.

If $b_1(X)=0$, then $SW^+(X,\mathfrak{s})-SW^-(X;\mathfrak{s})=\pm 1$. (Wall crossing formula)

Properties of the Seiberg-Witten invariants

Let's assume $b^+(X)>1$. $\mathrm{Spin}^c(X)=\{\text{spin}^c \text{ structures on } X\}$

Then we can view SW as a map $\mathrm{Spin}^c(X)\longrightarrow\mathbb{Z}$

Alternatively, we can view SW as a map

$SW_X: \mathrm{Char}(X)\to\mathbb{Z}$ $\qquad SW(\alpha) := \sum_{\substack{\mathfrak{s}\in\mathrm{Spin}^c(X)\\ c_1(\mathfrak{s})=K}} SW(X,\mathfrak{s})$

$\mathrm{Char}(X) = \{K\in H^2(X;\mathbb{Z}) \mid K\cdot[\Sigma]\equiv\Sigma\cdot\Sigma \bmod 2\}$

when $H_1(X;\mathbb{Z})$ has no 2-torsion (e.g. $\pi_1(X)=1$),

$c_1(\mathfrak{s})=c_1(\mathfrak{s}') \Rightarrow \mathfrak{s}=\mathfrak{s}'$. so this cause no loss of information.

We say $K\in\mathrm{Char}(X)$ is a basic class if $SW_X(K)\neq 0$

Basic properties of Seiberg-Witten invariants

$$SW : \mathrm{char}(X) \longrightarrow \mathbb{Z}$$

Assume $b_2^+(X) > 1$.

1) $SW_X(K) = 0$ for all but finitely many $K \in \mathrm{Char}(X)$

2) If X has a psc metric, then $SW_X(K) = 0 \quad \forall K$.

3) Conjuagtion symmetry $SW_X(-K) = (-1)^{b_2^+(X) - b_1(X) + 1} SW_X(K)$

4) Vanishing property If $X \cong_{\mathrm{diff}} X_1 \# X_2$ with $b_2^+(X_1), b^+(X_2) > 0$, then $SW_X(K) = 0 \quad \forall K$.

5) Blow-up formula If $X \cong_{\mathrm{diff}} X' \# \overline{\mathbb{CP}}^2$, then

basic classes of $X = \{ K \pm P.D.E \mid K$ is basic class of $X' \}$

More over $SW_X(K \pm P.D.(E)) = SW_{X'}(K)$.

6) If X is an algebraic surface. Then $SW_X(c_1(TX)) = \pm 1$.

7) If X is symplectic, let J be any compatible almost complex structure (i.e. $\omega(-, J-)$ gives a Riemannian metric). Then

$$SW_X(c_1(TX)) = \pm 1. \quad \text{(Taubes)}$$

8) Adjunction inequality

Let $\Sigma \hookrightarrow X$ be a smoothly embedded orientable surface $[\Sigma] \neq 0 \in H_2(X;\mathbb{R})$. Let K be a basic class. Then

i) If $[\Sigma]^2 \geq 0$, then $2g(\Sigma) - 2 \geq |\Sigma|^2 + |K.|\Sigma||$ (genus)

ii) If X is simple type (i.e. $SW_X(\mathfrak{s}) = 0$ unless $d(\mathfrak{s}) = 0$)

iii) and $g(\Sigma) \neq 0$, then $2g(\Sigma) - 2 \geq |\Sigma|^2 + |K.|\Sigma||$.

If $\Sigma = S^2$, then $|\Sigma|^2 < 0$.

Proof of basic properties:

(1) We have actually proved that $\bigsqcup_{\mathfrak{s} \in Spin^c(X)} \mathcal{M}_{SW}(X, \mathfrak{s})$ is compact.

so $\mathcal{M}_{SW}(X, \mathfrak{s}) \neq \phi$ only for finitely many $\mathfrak{s}$.

(2) Recall the C^0-bound $|\phi|^2_{C^0}(|\phi|^2_{C^0} + (\min_{p \in X} s(p) + |\eta|)) \leq 0$

If $s(p) > 0 \ \forall p$, we can choose η small enough s.t. $\phi = 0$ for any solutions (A, ϕ). However, by choosing generic η, no reducible solution exists. So $\mathcal{M}_{SW}(g, \eta)$ is empty.

(3) For any $\mathfrak{s} \in Spin^c(X)$ with $\rho: T_*X \to End(S)$

We can treat ρ as $T_*X \longrightarrow End(\bar{S})$ ← complex conjugate

This gives the conjugate $Spin^c$-structure $\bar{\mathfrak{s}}$, with $c_1(\bar{\mathfrak{s}}) = -c_1(\mathfrak{s})$.

$[(A,\phi)] \in \tilde{\mathcal{M}}_{SW}(\mathfrak{s},\eta) \Leftrightarrow [(A,\phi)] \in \tilde{\mathcal{M}}_{SW}(\bar{\mathfrak{s}},-\eta)$.

so $SW_X(K) = \pm SW_X(-K)$.

(4) This related to the gluing problem: Suppose $X = X_1 \cup_Y X_2$

How to compute SW_X in terms of SW_{X_1}, SW_{X_2}?

There is a whole package called "monopole Floer homology" to handle this for general Y.

In our case, $Y = S^3$ w/ psc, so the theory is simpler.

We consider the metric/perturbation

fixed, psc

$$(X, g_R, \eta) = (X_1^\circ, g_1, \eta_1) \cup ([-R,R]\times Y, [-R,R]\times g_Y, 0) \cup (X_2^\circ, g_2, \eta_2)$$

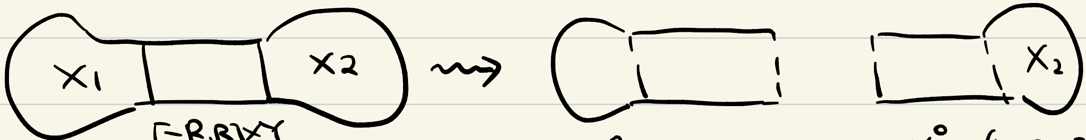
$X_1^\circ = X_1 \setminus D^4$, g_1 fixed, η_1 fixed, compact supported, similar for X_2°

$Y\times\{0\}$, base point

$$P_{X,R} = \widetilde{SW}^{-1}_{g_R}(\eta)/\mathcal{G}_{h,b}, \quad \mathcal{G}_{h,b} = \{\text{harmonic map } X \to S^1,\ b \mapsto 1\}$$

$$\mathcal{M}_{SW}(X, g_R) = P_{X,R}/S^1$$

Neck stretching: $R \to +\infty$



$X_1^\circ \cup [0,+\infty)\times Y = X_1^*$ $\qquad$ $X_2^\circ \cup (-\infty,0]\times Y = X_2^*$

solution $\rightsquigarrow$ solution $\qquad$ solution.

For $R \gg 0$, we have $P_{X,R} \cong P_{X_1^*} \times P_{X_2^*} \cong P_{X_1,R} \times P_{X_2,R}$ $\qquad$ $X_{i,R} = X_i^\circ \cup [0,R]\times Y \cup D^4$

Since $b_2^+(X_i)>0$, $P_{X_1,R}$, $P_{X_2,R}$ don't have reducible solutions
so S^1 acts freely

We have two actions of S^1 on $P_X = P_{X_1}\times P_{X_2}$

τ_1 : $e^{i\theta}(x_1,x_2) = (e^{i\theta}x_1, e^{i\theta}x_2)$

τ_2 : $e^{i\theta}(x_1,x_2) = (e^{-i\theta}x_1, e^{i\theta}x_2)$

Consider the pull-back diagram

$$\begin{tikzcd}
P_X \arrow[r, "-/\tau_2"] \arrow[d, "-/\tau_1"'] & P_X/\tau_2 \arrow[d, "-/\tau_1"] \\
P_X/\tau_1 \arrow[r, "-/\tau_2"] \arrow[d, phantom, "\shortparallel"] & P_X/_{\tau_1\times\tau_2} = P_{X_1}/S^1 \times P_{X_2}/S^1 = \mathcal{M}_{SW}(X_1)\times\mathcal{M}_{SW}(X_2) \\
\mathcal{M}_{SW}(X) &
\end{tikzcd}$$

$$\begin{aligned}
SW(X) &= c_1(P_X)^{\frac{d}{2}}\cdot[\mathcal{M}_{SW}(X)] \\
&= c_1(P_X/\tau_2)^{\frac{d}{2}}\cdot((-/\tau_2)_*[\mathcal{M}_{SW}(X)]) \\
&= 0
\end{aligned}$$
$\square$.

(Note $d(\mathfrak{s}_1 \# \mathfrak{s}_2) = d(\mathfrak{s}_1)+d(\mathfrak{s}_2)+1$. So if we only consider $SW(X,\mathfrak{s})$ with $d(\mathfrak{s})=0$, one of $d(\mathfrak{s}_1)$, $d(\mathfrak{s}_2)$ must <0. So the moduli space must be 0.

5) $X = X' \# \overline{\mathbb{CP}^2}$. consider $\mathfrak{s} = \mathfrak{s}' \# \mathfrak{s}_0$ with $c_1(\mathfrak{s}_0) = \pm P.D.(E)$.
(X': $\not\in$ PSC)

Then $P_{\overline{\mathbb{CP}^2}}$ = point, with trivial S^1-action.

so $P_X = P_{X'} \times P_{\overline{\mathbb{CP}^2}} = P_{X'}$. $\Rightarrow SW(X, \mathfrak{s}) = SW(X', \mathfrak{s}')$.

so $SW_X(K \pm P.D.(E)) = SW_X(K)$.

(More work is needed to show $SW_X(K \pm a \cdot P.D.(E)) = 0$ if $a > 1$).

6) Over algebraic surfaces, one can deform the S.W. equations so that solutions have algebraic geometrical meaning.

Given any almost complex structure $J: T_*X \to T_*X$, there is a "canonical spin$^{\mathbb{C}}$ structure" $\mathfrak{s}_J$ with

$S^+ = \Lambda^{0,0}(X) \oplus \Lambda^{0,2}(X) \qquad S^- = \Lambda^{0,1}(X)$

$= \underline{\mathbb{C}} \oplus \underbrace{K_X}_{\text{canonical line bundle}} \quad c_1(K_X) = -c_1(TX)$

$S^\pm$ has a spin$^{\mathbb{C}}$ connection A_0 coming from the Kahler metric.

One can show $(A_0, 1)$ is the unique solution.

So $SW_X(\mathfrak{s}_J) = \pm 1$.

7) The same strategy works. Taubes actually proves a much more general theorem "SW = Gr".

8) will prove next time.