A Thurstononian approach to the diffeomorphism group of a 4-manifold

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workshop

Joint work with Benson Farb (Univ. of Chicago)

M is a closed oriented manifold,

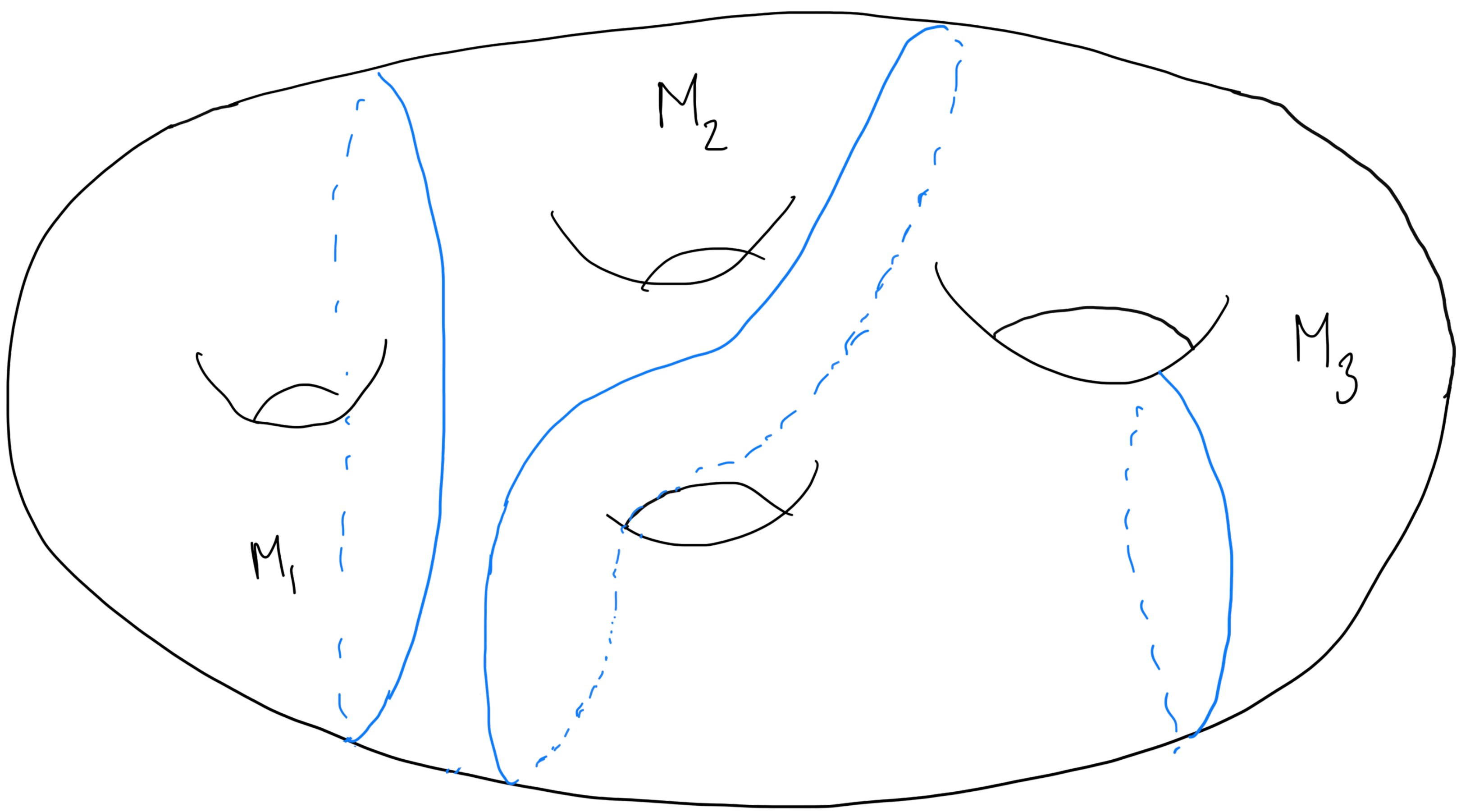
$$h : M \to M$$

an orientation preserving diffeo.

Question: Find `best' representative in its isotopy class [h].

Thurston's answer in dimension 2 (1970's)

The isotopy class [h] determines a finite collection of simple closed curves on M (up to isotopy) that are *essential* (none bounds a disk) and whose isotopy classes are permuted by h. They decompose M into connected surfaces M_i:



and a best representative h of [h] is such that th pos. power of h which preserves each M_i acts on M_i either with finite order or as a *pseudo-Anosov* map.

Answer for dimension 3 not so interesting (implicitly also due to Thurston)

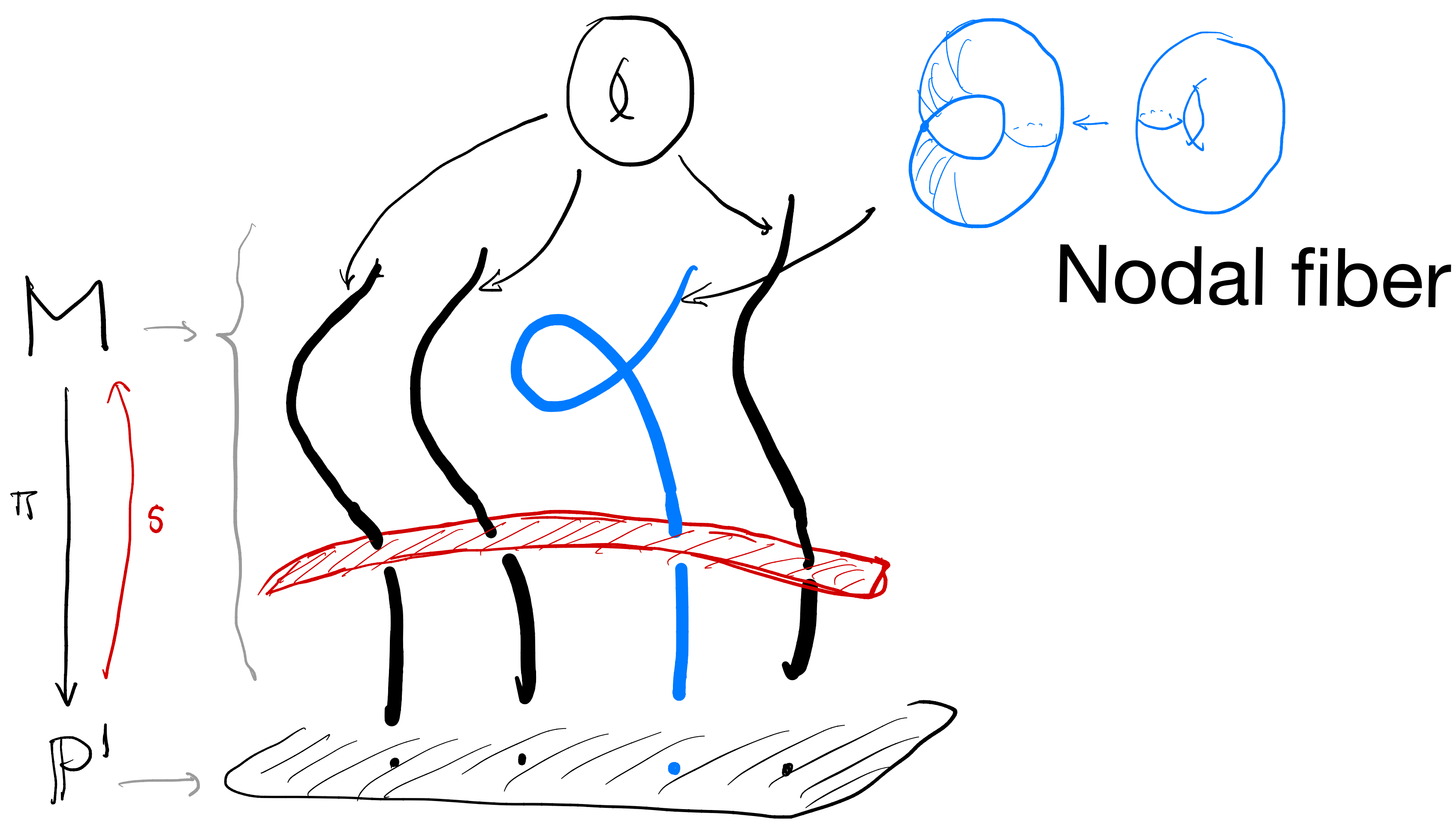
So let us go for dimension 4. Fundamental group introduces it own set of problems, so take M *simply connected* 4-manifold.

Theorem (Kreck, Perron, Quinn, Gabai et al.)

Two or. pres. homeomorphisms of M are isotopic if and only if they induce the same map on $H_2(M)$. All intersection form preserving automorphisms of $H_2(M)$ thus occur.

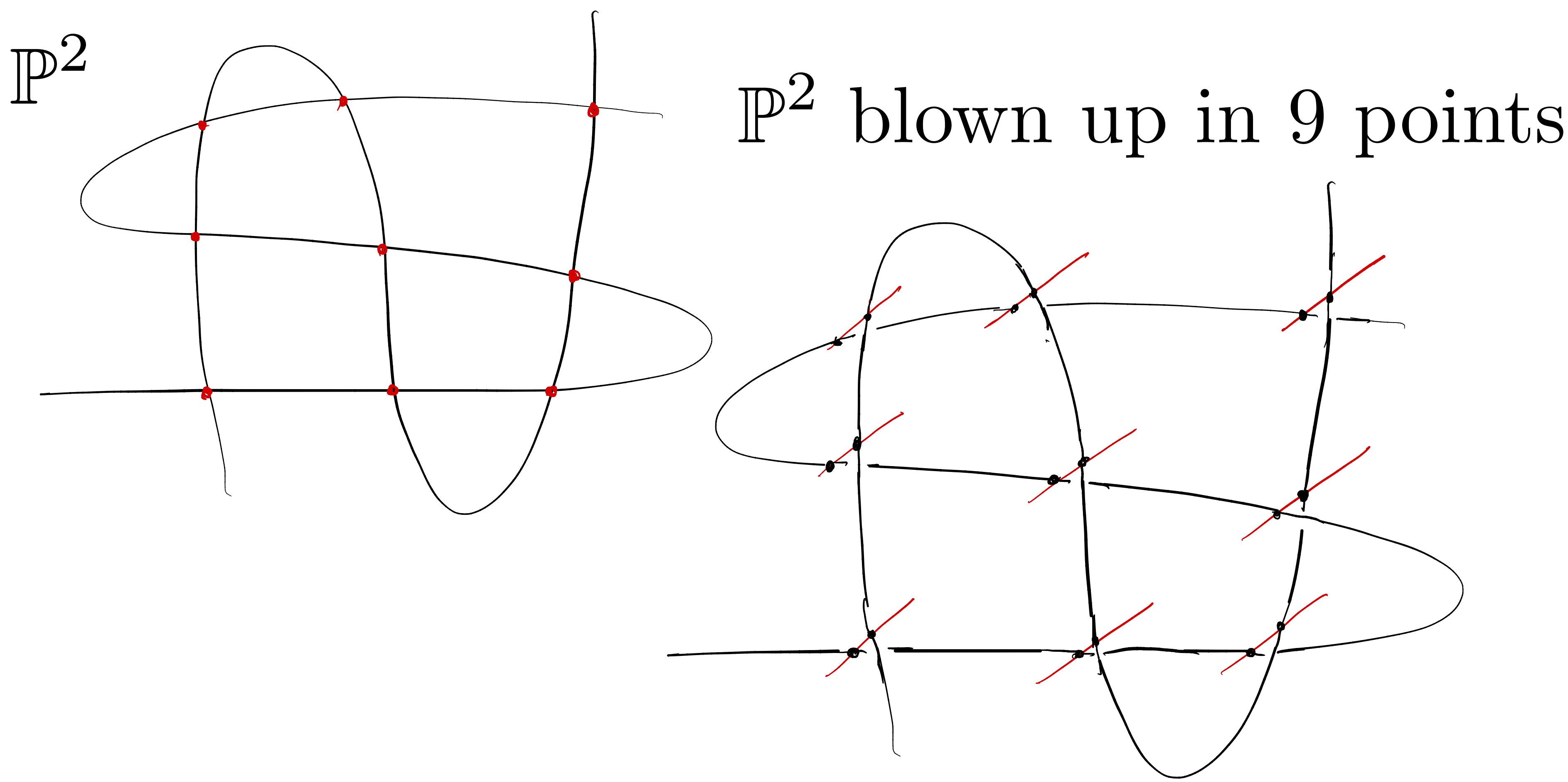
Theory of complex surfaces then offers a great class of examples: elliptic surfaces.

What is a (simply connected) elliptic surface?



Genus one fibration of M (allows simple type of singular fibers). We also assume it admits a section (if that is part of the data, we call it an elliptic fibration.

Example: Rational elliptic surface:
(generic) cubic pencil blown up in 9 points.



Construction gives 9 sections (but there many more).
It has exactly 12 nodal fibers (Euler char. computation)

Other examples from this one obtained by `base change':

Choose a holomorphic map of degree d>0, e.g.,

$$\mathbb{P}^1 \to \mathbb{P}^1, z \mapsto z^d$$

Then pull back the family along it and get

$$\pi_d : M_d \to \mathbb{P}^1$$

This family has 12d singular fibers and admits sections.

For d=2 we get a K3 surface; for d>3 we get simply connected surfaces of Kodaira dimension one (d is the arithmetic genus).

Theorem (Moishezon): Every simply connected elliptic surface is diffeomorphic to $\pi_d : M_d \to \mathbb{P}^1$ for some d.

Properties of $M_d \to \mathbb{P}^1$

Put $\Lambda_d := H_2(M_d)$

Let $e \in \Lambda_d$ be the fiber class

1. The intersection pairing on Λ_d is even/odd when d even/odd, unimodular and of signature (2d-1, 10d-1).

2. $e.e = 0$ and $\Lambda(e) := e^\perp/\mathbb{Z}e$ is even unimodular of sign $(2d-2, 10d-2)$ (such a lattice is unique up to isom.!)

3. (Friedman-Morgan) For d>2, every diffeomorphism of M_d preserves the fiber class up to sign.

Smooth Mordell-Weil group

Every smooth fiber is a flat torus: it has a translation group isomorphic to a torus. Something similar is true for the singular fibers. The diffeos that are fiberwise translations form a large group. It permutes the smooth sections simply transitively (two smooth sections differ by a fiberwise translation)

We call its group of connected components the Smooth Mordell-Weil group (must be abelian).

Theorem (Farb-L):

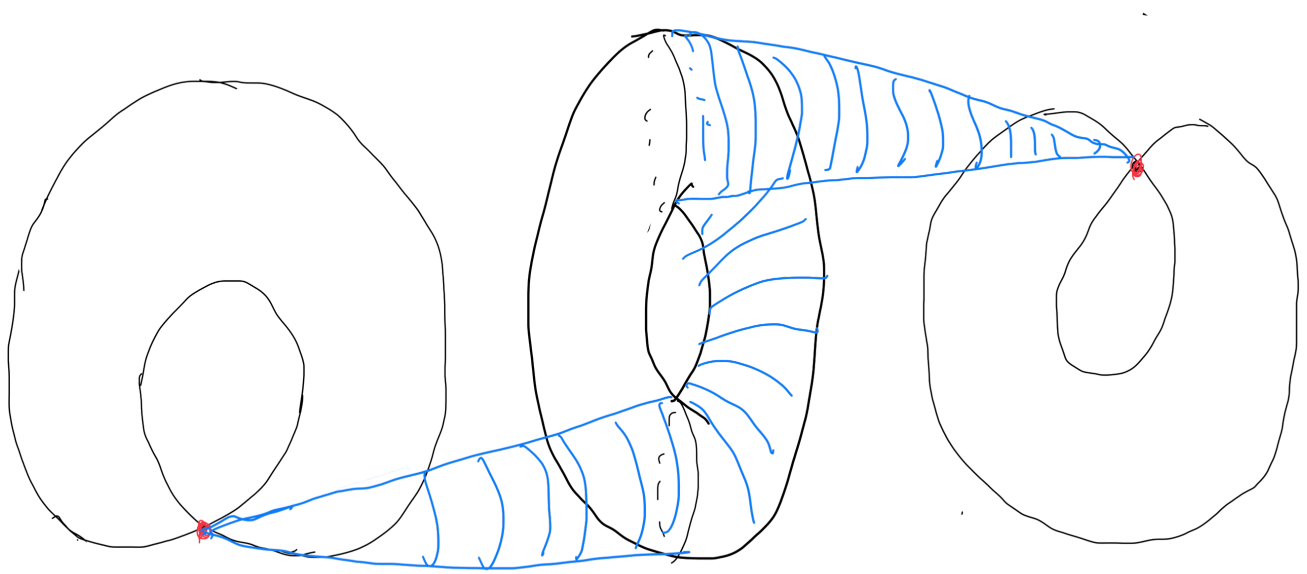
The smooth M-W group is acts faithfully on Λ_d as a group of Eichler transformations, making it isomorphic to $\Lambda_d(e) = e^{\perp}/\mathbb{Z}e$

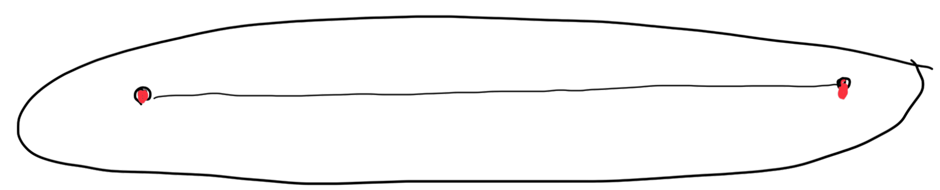
This group admits a `Nielsen realisation' as a group of fiberwise translations.

Theorem (Farb-L):

All the orthogonal transformations of Λ_d which preserve the fiber class can be realized by a fiber preserving diffeomorphism.

The construction is based on how *equinodal* fiber pairs determine 2-dimensional Dehn twists.

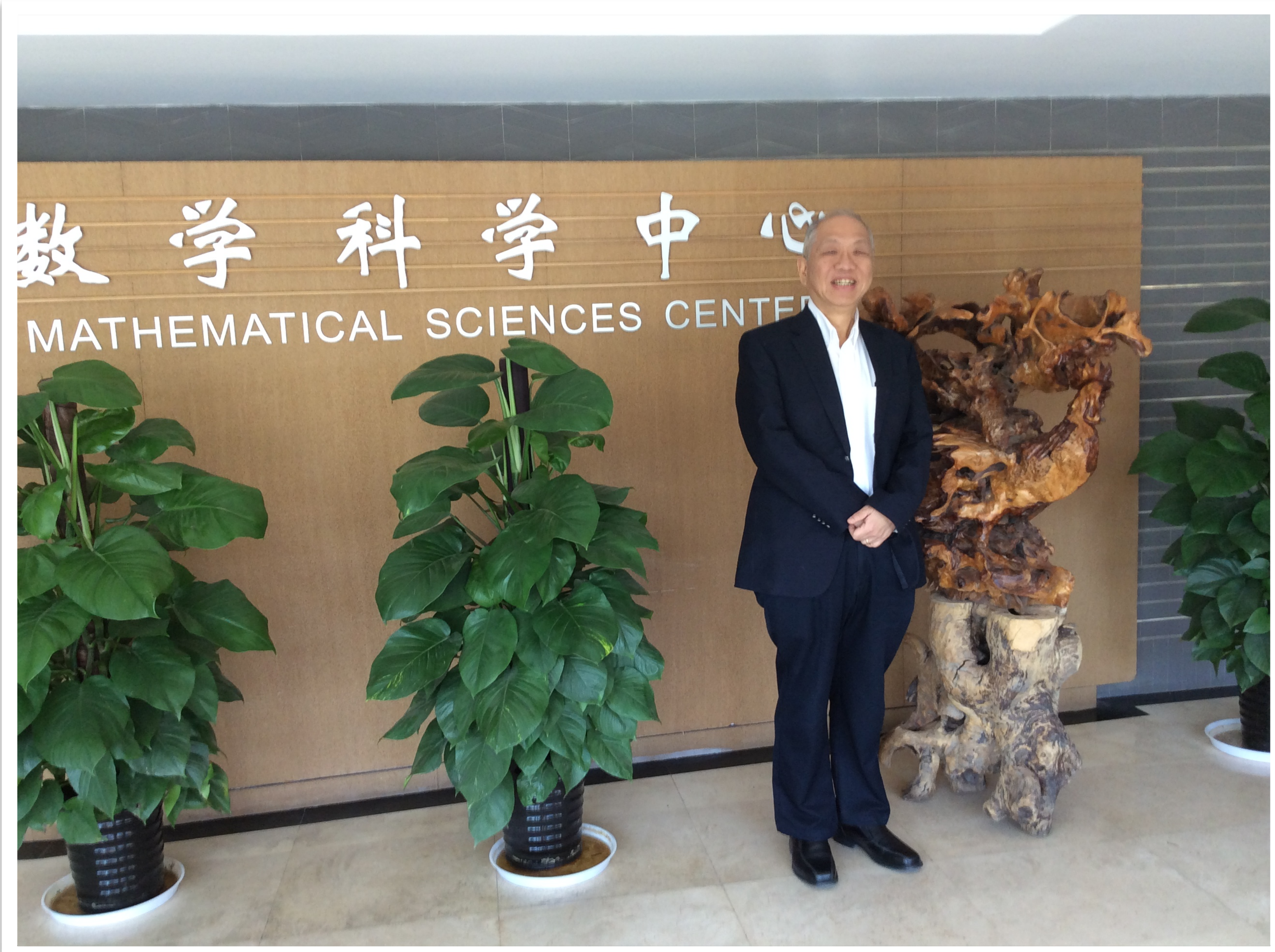




Two Lefschetz thimbles define 2-sphere (smooth!) with self-intersection -2. Two-dimensional Dehn twist defined by it can be made fiber preserving.

Corollary:

Assume d>2. Then every diffeo of M takes the elliptic fibration to one which is *topologically* isotopic to it: the top. isotopy class of this fibration is a diffeo invariant!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