Singular cscK metrics

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October 26, 2021

Goals and structures

Goals: to find metrics of **particular** curvature on manifolds.

Objects:

- Kähler metrics on complex manifolds
- Scalar curvature
- Cone singularities

Part 1: Brief review of some complex geometry and Kähler manifolds;

Part 2: Calabi's question on the existence of Kähler metrics with constant scalar curvature;

Part 3: Conical continuity method and Kähler cone metrics.

Uniformisation theorem of Riemann surfaces

A Riemann surface is a complex manifold with complex dim 1.

A Riemann surface has real dim 2.

1907: Every compact Riemann surface

- admits a metric of constant Gaussian curvature,
- unique up to holomorphic automorphisms.

positive curv   zero curv   negative curv (*genus* > 1)

Q: what is the uniformisation of higher-dimensional complex manifolds?

Gaussian curvature $- - - - - \rightarrow$ Scalar curvature

Local model of complex manifolds

Brief review of some complex geometry and Kähler manifolds:

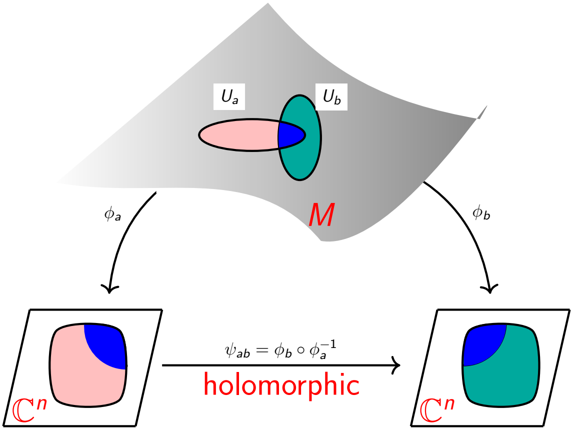
$$\mathbb{C} \xleftrightarrow{\textit{identified}} \mathbb{R}^2,$$
$$z = x + iy \mapsto (x, y),$$
$$i \longleftrightarrow J_E.$$

Linear complex structure J_E is an endormorphism of $\mathbb{R}^2$:

$$J_E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ s.t. } J_E^2 = -id.$$

We think $\mathbb{C}^n$ as a direct sum of $\mathbb{C}$.

Complex manifolds with complex dimension n

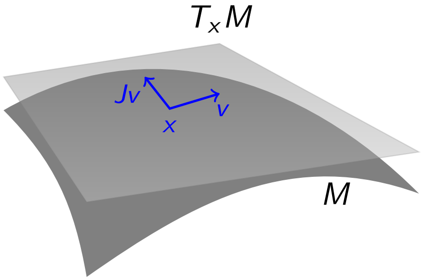


A function $F : U \in \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic if J_E commutes with the Jacobian matrix of F, that is $J_E \circ DF = DF \circ J_E$.

Complex structures

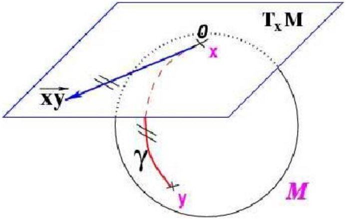
Pulling back J_E to TM.

Complex structure $J : TM \to TM, \quad J^2 = -id.$



Riemannian metrics

- Riemannian metric $g(X, Y), \forall X, Y \in TM$.
- angles, lengths, distances
- curvatures...



Example: Euclidean metric:

- $g(X, Y) = (X, Y)$.
- The Euclidean distance $d(p, q) = \sqrt{(p - q, p - q)}$.

What is a Kähler manifold M of complex dim n?

The associated form of g is defined to be:

$$\omega(X, Y) := g(JX, Y).$$

Definition

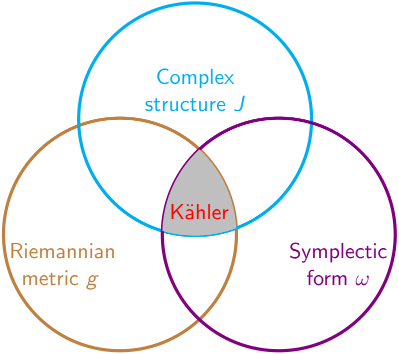
A Kähler manifold is a complex manifold with closed associated form ω, that is

$$d\omega = 0.$$

$$d\omega = 0 \Longleftrightarrow \omega \text{ is a } \textbf{symplectic structure}.$$

Kähler manifolds

Compatibility: $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$.



The concept was introduced by Erich Kähler in 1933 in his article "Über eine bemerkenswerte Hermitesche Metrik".

Erich Kähler (1933)
from Wikipedia

Kähler manifolds

Examples:

- in complex dim 1,
  every Riemann surface is Kähler;
- in complex dim 2,
  Kähler $\Longleftrightarrow$ the first Betti number is even (Siu 1980's);
  Fails for $n \geq 3$ (Hironaka example 1962);
- every complex projective variety is Kähler...

  "$\Leftarrow$" the converse direction:
  the cohomology class defined by ω is integral $\Rightarrow$ projective (Kodaira 1954).

Metrics on Kähler manifolds

Kähler (Symplectic) condition: $d\omega = 0$

$$\Downarrow$$

ω determines a cohomology class,

known as the Kähler class $[\omega]$.

$$\Downarrow$$

Any other Kähler metrics in $[\omega]$ can be

represented by a potential function φ,

$$\omega_\varphi := \omega + i\partial\bar{\partial}\varphi.$$

Simple explicit expressions for curvature tensors

Recall the Kähler metric in local co-ordinates z^a,

$$g_\varphi = \sum_{a,b}(g_{a\bar{b}} + \frac{\partial^2\varphi}{\partial z^a \partial z^{\bar{b}}})dz^a \otimes dz^{\bar{b}}.$$

- Ricci curvature is given by

$$Ric_{c\bar{d}}(g_\varphi) = -\frac{\partial^2}{\partial z^c \partial z^{\bar{d}}} \log\det(g_\varphi).$$

- Scalar curvature is the **trace** of the Ricci curvature:

$$S(\varphi) = \sum_{c,d} g_\varphi^{c\bar{d}} Ric_{c\bar{d}}(g_\varphi) = -\triangle_\varphi \log\det(g_\varphi).$$

S is a function involving the 4th derivatives of φ.

Calabi's CSCK problem

Question (Calabi 1950s)

To find the constant scalar curvature Kähler metrics (cscK metrics) in a given Kähler class $[\omega]$ satisfying

$$\textit{the nonlinear PDE: } S(\varphi) = constant.$$

It has been answered affirmatively in the following cases:

1. the uniformisation theorem (n=1, 1907);
2. Kähler-Einstein metrics $Ric = \lambda\omega$.

In general Kähler class, it is a hard question.

Kähler-Einstein metrics

Kähler-Einstein metrics satisfy

$$Ric = \lambda\omega, \text{ for } \lambda = -1, \quad 0, \quad 1.$$

They are all cscK, but satisfy 2nd Monge-Ampère equations

$$\det(g_\varphi) = e^{-\lambda\varphi}.$$

- $\lambda = 0$, Calabi-Yau metric (Yau 1970s);
  $\rightsquigarrow$ Mathematical physics
- $\lambda = -1$, Aubin and Yau independently (1970s).
- $\lambda = 1$, the Fano KE case is more subtle;

CscK problem

In general Kähler class,

Conjecture (Yau-Tian-Donaldson conjecture)

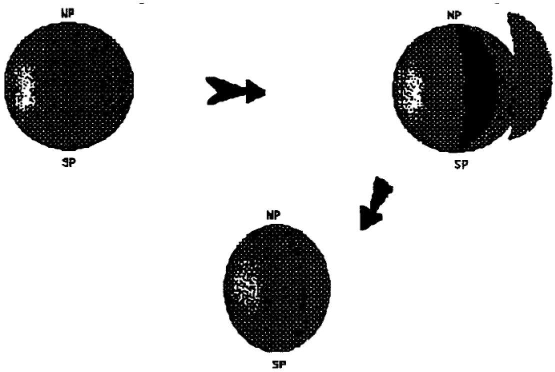
$$\textit{cscK metrics} \iff \textit{K-stability}.$$

Continuity method:

1. Starting point;
2. Openess;
3. Closedness.

The limit of a sequence of smooth Kähler metrics develops singularities $\rightarrow$ algebraic structure $\rightarrow$ K-stability.

A toy model of cscK cone metric



European football $\Longrightarrow$ American football with two cone singularities NP and SP

Conical continuity method

Definition

A Kähler cone metric ω of cone angle $2\pi\beta$ along a divisor D satisfies that

- it is a smooth Kähler metric on $M \setminus D$,
- it is quasi-isometric to the model cone metric near D

$$\omega_{cone} = |z^1|^{2(\beta-1)} idz^1 \wedge dz^{\bar{1}} + \sum_{2\leq a\leq n} idz^a \wedge dz^{\bar{a}}.$$

Here z^1 is the local defining functions of D.

How to define cscK cone metrics?

Definition

We say ω is a cscK cone metric, if

- it is a Kähler cone metric;
- it has constant scalar curvature on $M \setminus D$.

New approach to the cscK problem.

Q: how to formulate the geometric definition into PDEs?

Naive idea: $S = \underline{S} + 2\pi(1-\beta)\mathrm{tr}_{\varphi}[\mathrm{D}]$.

Unfortunately, the red part is **NOT** well-defined!

New idea: reduce it to a couple of PDEs. (Z. 2017)

The conical continuity method

Plan: to apply the conical continuity method to the cscK problem!

Conical continuity method deforming cone angle of the cscK cone metrics, leads to the existence of smooth cscK metrics.

Sketch of the project:

1. Small angle cscK cone metrics (PLMS, Math Ann 2018);
2. openess: show that given one solution the cone angle can be slightly deformed (CPAM 2018);
3. approximation; show that the cone metric is approximated by smooth metrics (arXiv:1803.09506);
4. a priori estimates and closedness (arXiv:1805.04944).
5. Log K-Stability (arXiv:2110.02518)

Conical continuity method approach to the cscK problem

The cscK cone path is a family of cscK cone metrics

$$"S = \underline{S} + 2\pi(1-\beta)\mathrm{tr}_\varphi[\mathrm{D}]".$$

Theorem (Z. 2018)

The cscK cone path has two properties:
openness and approximation.

The proof requires the theory of cscK cone metrics including

- sharp regularity in correct weighted spaces,
- uniqueness,
- existence.

Difficulties: large angles

$$\omega_D = \omega + \delta \frac{\sqrt{-1}}{2} \partial\bar{\partial}|s|^{2\beta} \text{ is a Kähler cone metric.}$$

We compute the geometric quantities of ω_D, under the coordinates

$$w^1 = |z^1|^{\beta-1} z^1, \quad z^2, \quad \cdots, \quad z^n.$$

- $\log \omega_D^n$ is bounded for $0 < \beta \leq 1$.
- $\nabla^1 \omega_D$ is bounded for $0 < \beta < \frac{2}{3}$.
- $\nabla^2 \omega_D$ is bounded for $0 < \beta < \frac{1}{2}$.
- In order to have bounded $\nabla^k \omega_D$ $k = 3, 4, 5, \cdots$, the angle has to be smaller and smaller...

We need to introduce NEW weighted Hölder spaces!

Regularity

Theorem (Regularity of cscK cone metrics, Z. 2017)

The Kähler potential of a cscK cone metric is geometrically polyhomogeneous.

Let φ be the Kähler potential of a cscK cone metric.

PDEs of cscK cone metrics

$\Downarrow$

Determine the growth rates of ALL derivatives of φ

(A priori estimates)

$\Downarrow$

the asymptotic expansion of φ

$\Downarrow$

New geometric definitions of weighted Hölder spaces

Uniqueness

Theorem (Uniqueness of cscK cone metrics, Z. 2017)

The cscK cone metric is unique up to holomorphic automorphisms.

Previous results:

- Riemannian surface (Thurston, McOwen, Troyanov 1970s-80s);
- Smooth cscK metrics (Berman-Berndtsson 2014, Chen-Păun-Zeng 2015);
- KE metrics (BM), KR solitons (Zhu);
- KE cone metrics ...

Uniqueness implies log K-semistability

Question (Log cscK problem)

CscK cone metrics $\Longleftrightarrow$ *log K-stability?*

Theorem (AHZ 2021)

The existence of a cscK cone metric $\Longrightarrow$ *log K-semistability.*

Previous results:

- smooth cscK metrics (Donaldson 2005);
- orbifold cscK metrics (Ross-Thomas 2011).

Properness

Definition

We say a energy functional F is proper, if for any sequence $\{\varphi_i\} \subset \mathcal{H}$, we have

$$\lim_{i\to\infty} d_{1,G}(0,\varphi_i) = \infty \implies \lim_{i\to\infty} F(\varphi_i) = \infty.$$

- The properness condition is motivated from the Moser-Trudinger inequality on S^2 (Ding-Tian 1988).
- Properness $\iff$ existence of Fano KE metrics (Tian 1997)
- Properness $\iff$ cscK metrics $\iff$ geodesic stability (Chen-Cheng 2018).

Existence

Theorem (Log properness theorem, Z. 2018)

CscK cone metrics $\iff$ properness of the log K-energy $\iff$ log geodesic stability.

Theorem (AHZ 2021)

CscK cone metrics $\Rightarrow$ *log K-stability*.

Further questions

Both the space M and the metric are singular!

Q: What kind of singularities are allowed such that the singular cscK metric is unique up to automorphisms?

Q: What about existence?

$\rightsquigarrow$ Singularity models of the cscK problem

Singular cscK metrics with possible degeneration

Let Ω be a real $(1,1)$-cohomology class.

Definition

A class Ω is *nef*, if it lies in the closure of the Kähler cone. That is the cohomology class $\Omega + t[\omega_K]$ is Kähler for all $t > 0$.

Definition

We say a class Ω is *big*, if it contains a *Kähler current*, i.e. a closed positive $(1,1)$-current T satisfying $T \geq t\omega_K$ for some $t > 0$.

The definition of singular cscK metric generalises the notion of singular KE metric.
The singular KE metric is widely studied in the minimal model program from birational geometry and there are many literatures on finding singular KE metrics on a minimal projective manifold of general types or log Fano varieties, such as many beautiful works including BBEGZ, BBGZ, BEGZ, EGZ, Song-Tian...

Singular cscK metrics with possible degeneration

Proposition

Let Ω be a big and nef class on a Kähler manifold (X, ω_K), whose automorphism group $Aut(M)$ is trivial. Suppose that Ω satisfies the following cohomology condition (1),

$$\begin{cases} (i) & 0 \leq \eta < \frac{n+1}{n}\alpha_\beta, \\ (ii) & C_1(X, D) < \eta\Omega, \\ (iii) & (-n\frac{C_1(X,D)\cdot\Omega^{n-1}}{\Omega^n} + \eta)\Omega + (n-1)C_1(X, D) > 0. \end{cases} \tag{1}$$

Then Ω has the cscK approximation property.
Precisely, setting $\Omega_t = \Omega + t[\omega_K]$, we obtain that for $0 < t \leq \tilde{t}$,

1. *the log K-energy is J-proper in Ω_t,*
2. *there exists a cscK cone metric ω_t in Ω_t,*
3. *the cscK cone metric ω_t a smooth approximation $\omega_{t,\epsilon}$ in Ω_t.*

Normal complex projective variety

We set the pair $(Y, \triangle)$ consist of a connected normal complex projective variety Y and a Weil Q-divisor $\triangle$. Assume $K_Y + \triangle$ is Q-Cartier, that is there is a positive integer r such that $r(K_Y + \triangle)$ is Cartier.

Definition

A *log resolution* $\pi : X \to Y$ of $(Y, \triangle)$ gives

$$K_X = \pi^*(K_Y + \triangle) + D.$$

In which, $a_i \in \mathbb{Q}$ is called the *discrepancy* of $(Y, \triangle)$ along E_i. Actually,

- $D := \sum_i a_i E_i$ is a $\mathbb{Q}$-divisor and $\cup_i E_i$ has normal crossing,
- $\pi_* D = -\triangle$.

Singular cscK metrics

Definition

We say $\omega_\varphi = \omega_K^Y + i\partial\bar{\partial}\varphi$ is a singular cscK metric in a Kähler class $\Omega = [\omega_K^Y]$ on a normal complex space Y, if

- $\varphi \in PSH(\omega_K^Y)$,
- φ is a L^1-limit of a sequence of smooth cscK potentials.

Similarly, we say is ω_φ is a bounded singular cscK metric, if φ is $\mathcal{E}^1(\omega_K^Y) \cap L^\infty$.

We could say more about the control of the 2nd estimates, e.g.

$$\int_X (\mathrm{tr}_{\omega_{\mathbf{b}_\epsilon}} \omega_{\varphi_\epsilon})^{\mathrm{p}} |\mathrm{s}|^\sigma_{\mathrm{h_E}} \omega^{\mathrm{n}}_{\mathbf{b}_\epsilon} \leq \mathrm{C}, \quad \forall \mathrm{p} \geq 1.$$

Example: log Fano pairs

We set $\triangle$ to be an effective Q-divisor. The divisor D has two parts. One part is with coefficients in $(-1,0]$. Then by definition, the cone angle β_i is equal to $1+\alpha_i$. So we have $\beta \in (0,1]$.
The other part is the π-exceptional divisor, which is effective and with integer coefficients.

Definition

The pair $(Y,\triangle)$ is *Kawamata log terminal (klt)*, if $a_i > -1$ for all i. When $\triangle = 0$, Y is said to be *log terminal*, if $(Y,0)$ is klt.

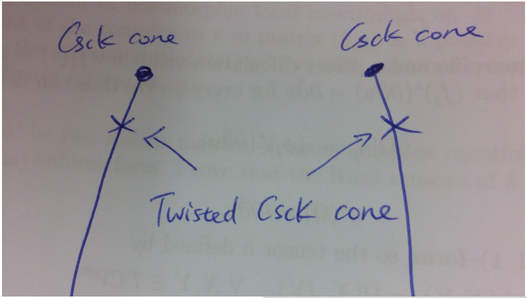
Definition

A klt pair $(Y,\triangle)$ is called a *log Fano pair*, if $-(K_Y+\triangle)$ is ample.

Assume that uniqueness is NOT true and there exist two different cscK cone metrics.

PART A:
we deform these cscK cone metrics to get two DIFFERENT twisted cscK cone metrics satisfying

$$S(\omega_{\varphi(t)}) - \underline{S}_\beta = (1-t)\left(\frac{\omega_0^n}{\omega_{\varphi(t)}^n} - 1\right). \tag{2}$$



Sketch of the proof of the uniqueness theorem

Proof of PART A:
we solve the linearised equation Lic at $t = 1$.

Regularity of (twisted) cscK cone metrics

(Weighted function spaces)

$+$

Structure of the automorphism group

(Kernel of Lic)

$\Downarrow$

Prove the Fredholm alternative for Lic

$\Downarrow$ Implicit function theorem

Construct the twisted cscK cone metrics (2)

for all $t \in (1-\tau, 1]$.

Sketch of proof of uniqueness theorem

PART B:

Construct cone geodesics in the space of Kähler cone metrics.

$$\Downarrow$$

Show that the log-K-energy is strictly convex along the cone geodesics.

$$\Downarrow$$

Conclude that the twisted cscK cone metric (2) is unique.

PART A contradicts to PART B.

Uniqueness is proved!

Thank you for your attention!