Combinatorics

Instructor: Prof. Jie Ma

Combinatorics 2022, Tsinghua

- This class notes will be updating throughout this course.
- The course website can be found at https://ymsc.tsinghua.edu.cn/info/1050/2595.htm

Contents

1 Enumeration

First we give some standard notation that will be used throughout this course.

- Let n be a positive integer. We will use $[n]$ to denote the set $\{1, 2, ..., n\}$.
- Given a set X, let $|X|$ denote the size of X, that is the number of elements contained in X.
- We use "#" to express the word "number".
- The *factorial* of n is the product

$$n! = n \cdot (n-1) \cdots 2 \cdot 1,$$

which can be extended to all non-negative integers by letting $0! = 1$.

1.1 Binomial Coefficients

Let X be a set of size n. Define $2^X = \{A : A \subseteq X\}$ to be the family of all subsets of X. Since the size of 2^X is equal to the number of binary vectors of length $|X|$ or the number of functions from X to $\{0, 1\}$, we have $|2^X| = 2^{|X|} = 2^n$.

Let $\binom{X}{k} = \{A : A \subseteq X, |A| = k\}$, we will use $\binom{n}{k}$ to denote $|\binom{X}{k}|$. For $n < k$, we know that $\binom{n}{k} = 0$ by definition.

Fact 1.1. *For integers $n > 0$ and $0 \leq k \leq n$, we have $|\binom{X}{k}| = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.*

Proof. If $k = 0$, then it is clear that $|\binom{X}{0}| = |\{\emptyset\}| = 1 = \binom{n}{0}$. Now we consider $k > 0$. Let

$$(n)_k := n(n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

First we will show that number of ordered k-tuples $(x_1, x_2, \ldots, x_k)$ with distinct $x_i \in X$ is $(n)_k$. There are n choices for the first element x_1. When $x_1, \ldots, x_i$ is chosen, there are exactly $n - i$ choices for the element x_{i+1}. So the number of ordered k-tuples $(x_1, x_2, \ldots, x_k)$ with distinct $x_i \in X$ is $(n)_k$. Since any subset $A \in \binom{X}{k}$ corresponds to $k!$ ordered k-tuples, it follows that $|\binom{X}{k}| = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}$. This finishes the proof. ∎

Next we discuss more properties of binomial coefficients.

Fact 1.2. (1). $\binom{n}{k} = \binom{n}{n-k}$ *for $0 \leq k \leq n$.*
(2). $2^n = \sum_{0 \leq k \leq n} \binom{n}{k}$.
(3). $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. *(Pascal's identity)*

Proof. (1) is trivial. Since $2^{[n]} = \cup_{0 \leq k \leq n} \binom{[n]}{k}$, we see $2^n = \sum_{0 \leq k \leq n} \binom{n}{k}$, proving (2). Finally, we consider (3). Note that the first term on the right hand side $\binom{n-1}{k-1}$ is the number of k-sets containing a fixed element, while the second term $\binom{n-1}{k}$ is the number of k-sets avoiding this element. So their summation gives the total number of k-sets in $[n]$, which is $\binom{n}{k}$. This finishes the proof. ∎

Pascal's triangle is a triangular array constructed by summing adjacent elements in preceding rows. By Fact 1.2 (3), in the following graph we have that the k-th element in the n row is $\binom{n}{k-1}$.

$$
\begin{array}{ll}
1 \cdots\cdots & 1 \quad 1 \\
2 \cdots\cdots & 1 \quad 2 \quad 1 \\
3 \cdots\cdots & 1 \quad 3 \quad 3 \quad 1 \\
4 \cdots\cdots & 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
5 \cdots\cdots & 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
6 \cdots\cdots & 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
7 \cdots\cdots & 1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
8 \cdots\cdots & 1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
9 \cdots\cdots & 1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1 \\
10 \cdots\cdots & 1 \quad 10 \quad 45 \quad 120 \quad 210 \quad 252 \quad 210 \quad 120 \quad 45 \quad 10 \quad 1
\end{array}
$$

Fact 1.3. *The number of integer solutions* $(x_1, \dots, x_n)$ *to the equation* $x_1 + \cdots + x_n = k$ *with each* $x_i \in \{0, 1\}$ *is* $\binom{n}{k}$.

Fact 1.4. *The number of integer solutions* $(x_1, \dots, x_n)$ *to the equation* $x_1 + \cdots + x_n = k$ *with each* $x_i > 0$ *is* $\binom{k-1}{n-1}$.

Proof. This question is equivalent to ask: How many ways are there of distributing k sweets to n children such that each child has at least one sweet.

Lay out the sweets in a single row of length k, and cut it into n pieces. Then give the sweets of the i_{th} piece to child i, which means that we need $n-1$ cuts from $k-1$ possibles. ∎

Fact 1.5. *The number of integer solutions* $(x_1, \dots, x_n)$ *to the equation* $x_1 + \cdots + x_n = k$ *with each* $x_i \geq 0$ *is* $\binom{n+k-1}{n-1}$.

Proof 1. Let $A = \{$integer solutions $(x_1, \dots, x_n)$ to $x_1 + \cdots + x_n = k$, $x_i \geq 0\}$ and $A = \{$integer solutions $(y_1, \dots, y_n)$ to $y_1 + \cdots + y_n = n + k$, $y_i > 0\}$. Then $|B| = \binom{n+k-1}{n-1}$ by Fact 1.4.

Define $f : A \to B$, by $f((x_1, \dots, x_n)) = (x_1 + 1, \dots, x_n + 1)$. It suffices to check that f is a bijection, which we omit here.

∎

Proof 2. Suppose we have k sweets (of the same sort), which we want to distribute to n children. In how many ways can we do this? Let x_i denote the number of sweets we give to the i-th child, this question is equivalent to that state above.

We lay out the sweets in a single row of length r and let the first child pick them up from left to right (can be 0). After a while we stop him/her and let the second child pick up sweets, etc. The distribution is determined by the specifying the place of where to start a new child. This is equal to select $n-1$ elements from $n+r-1$ elements to be the child, others be the sweets (the first child always starts at the beginning). So the answer is $\binom{n+k-1}{n-1}$. ∎

Exercise 1.6. *Let* $X = [n]$, $A = \{(a_1, a_2, \dots, a_r) | a_i \in X , 1 \leq a_1 \leq a_2 \leq \cdots \leq a_r \leq n, a_{i+1} - a_i \geq k + 1, i \in [r-1]\}$. *Prove that* $|A| = \binom{n-k(r-1)}{r}$.

Exercise 1.7. *Give a Combinatorial proof of*

$$\sum_{k=0}^{n} \binom{n}{k}\binom{k}{m} = \binom{n}{m} 2^{n-m}.$$

Exercise 1.8. *Give a Combinatorial proof of*

$$\sum_{k=0}^{m} \binom{m}{k}\binom{n+k}{m} = \sum_{k=0}^{m} \binom{n}{k}\binom{m}{k} 2^{k}.$$

1.2 Counting Mappings

Define X^Y to be the set of all functions $f : Y \to X$.

Fact 1.9. $|X^Y| = |X|^{|Y|}$.

Proof. Let $|Y| = r$. We can view X^Y as the set of all strings $x_1 x_2 \cdots x_r$ with elements $x_i \in X$, indexed by the r elements of Y. So $|X^Y| = |X|^{|Y|}$. ∎

Fact 1.10. *The number of injective functions $f : [r] \to [n]$ is $(n)_r$.*

Proof. We can view the injective function f as an ordered k-tuple $(x_1, x_2, \ldots, x_r)$ with distinct $x_i \in X$, so the number of injective functions $f : [r] \to [n]$ is $(n)_r$. ∎

Definition 1.11 (The Stirling number of the second kind). *Let $S(r, n)$ be the number of partitions of $[r]$ into n unordered non-empty parts.*

Exercise 1.12. *Prove that*

$$S(r, 2) = \frac{2^r - 2}{2} = \frac{1}{2} \sum_{i=1}^{r-1} \binom{r}{i}.$$

Fact 1.13. *The number of surjective functions $f : [r] \to [n]$ is $n!S(r, n)$.*

Proof. Since f is a surjective function if and only if for any $i \in [n]$, $f^{-1}(i) \neq \emptyset$ if and only if $\cup_{i \in [n]} f^{-1}(i) = [r]$, and $S(r, n)$ is the number of partition of $[r]$ into n unordered non-empty parts, we have the number of surjective functions $f : [r] \to [n]$ is $n!S(r, n)$. ∎

We say that any injective $f : X \to X$ is a **permutation** of X (also a bijection). We may view a permutation in two ways: (1) it is a bijective from X to X. (2) a reordering of X.

Cycle notation describes the effect of repeatedly applying the permutation on the elements of the set. It expresses the permutation as a product of cycles; since distinct cycles are disjoint, this is referred to as "decomposition into disjoint cycles".

Definition 1.14 (The Stirling number of the first kind). *Let $s(r, n)$ be the number of permutations of $[r]$ with exactly n cycles multiplied by $(-1)^{(r-n)}$.*

The following fact is a direct consequence of Fact 1.10.

Fact 1.15. *The number of permutations of $[n]$ is $n!$.*

Exercise 1.16. *(1) Let $S(r, n) = \left\{ {r \atop n} \right\}$, give a Combinatorial proof of $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$.*

(2) Let $s(n, k) = (-1)^{n-k} \left[{n \atop k} \right]$, give a Combinatorial proof of $\left[{n \atop k} \right] = \left[{n-1 \atop k-1} \right] + (n-1) \left[{n-1 \atop k} \right]$.

1.3 The Binomial Theorem

Define $[x^k]f$ to be the coefficient of the term x^k in the polynomial $f(x)$.

Fact 1.17. *For $j = 1, 2, \ldots, n$, let $f_j(x) = \sum_{k \in I_j} x^k$ where I_j is a set of non-negative integers, and let $f(x) = \prod_{j=1}^n f_j(x)$. Then, $[x^k]f$ equals the number of solutions $(i_1, i_2, \ldots, i_n)$ to $i_1 + i_2 + \cdots + i_n = k$, where $i_j \in I_j$.*

Fact 1.18. *Let $f_1, \ldots, f_n$ be polynomials and $f = f_1 f_2 \cdots f_n$. Then,*

$$[x^k]f = \sum_{i_1+\cdots+i_n=k, i_j \geq 0} \left(\prod_{j=1}^{n} [x^{i_j}]f_j \right).$$

Theorem 1.19 (The Binomial Theorem)**.** *For any real x and any positive integer n, we have*

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.$$

Proof 1. Let $f = (1+x)^n$. By Fact 1.17 we have $[x^k]f$ equals the number of solutions $(i_1, i_2, ..., i_n)$ to $i_1 + i_2 + \cdots + i_n = k$ where $i_j \in \{0, 1\}$, so $[x^k]f = \binom{n}{k}$. ∎

Proof 2. By induction on n. When $n = 1$, it is trivial. If the result holds for $n - 1$, then $(1+x)^n = (1+x)(1+x)^{n-1} = (1+x)\sum_{i=0}^{n-1} \binom{n-1}{i} x^i = \sum_{i=1}^{n-1} (\binom{n-1}{i} + \binom{n-1}{i-1}) x^i + 1 + x^n$. Since $\binom{n-1}{i} + \binom{n-1}{i-1} = \binom{n}{i}$ and $\binom{n}{0} = \binom{n}{n} = 1$, we have $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$. ∎

Fact 1.20. $\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2 = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$.

Proof 1. Since $(1+x)^{2n} = (1+x)^n(1+x)^n$, by Fact 1.18, we have $\binom{2n}{n} = [x^n](1+x)^{2n} = \sum_{i=0}^{n} ([x^i](1+x)^n)([x^{n-i}](1+x)^n) = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n} \binom{n}{i}^2$. ∎

Proof 2. (It is easy to find a combinatorial proof.) ∎

Exercise 1.21 (Vandermonde's Convolution Formula)**.**

$$\binom{n+m}{k} = \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \sum_{i+j=k} \binom{n}{i}\binom{m}{j}.$$

Exercise 1.22.

$$\binom{n+m}{r+m} = \sum_{i-j=r} \binom{n}{i}\binom{m}{j}.$$

Exercise 1.23. *Prove that*

$$\sum_{k=0}^{m} \binom{m}{k}\binom{n+k}{m} = \sum_{k=0}^{m} \binom{n}{k}\binom{m}{k} 2^k.$$

by Binomial Theorem.

Fact 1.24. (1).

$$\sum_{\textit{all even } k} \binom{n}{k} = \sum_{\textit{all odd } k} \binom{n}{k} = 2^{n-1}.$$

(2).

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Proof. (1). We see that $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$. Taking $x = 1$ and $x = -1$, we have

$$\sum_{\text{all even } k} \binom{n}{k} = \sum_{\text{all odd } k} \binom{n}{k} = 2^{n-1}.$$

(2). Let $f(x) = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. Then $f'(x) = n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} x^{k-1}$. Let $x = 1$, then we have $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$. ∎

Definition 1.25. *Let $k_j \geq 0$ be integers satisfying that $k_1 + k_2 + \cdots + k_m = n$. We define*

$$\binom{n}{k_1, k_2, \cdots, k_m} := \frac{n!}{k_1! k_2! \cdots k_m!}.$$

- When $m = 2$, $\binom{n}{k_1, k_2} = \binom{n}{k_1}$ is the number of binary vectors of length n with k_1 zero and k_2 ones, which is also the number of ordered partitions of $[n]$ into 2 parts such that the i_{th} part has size k_i.
- When $m \geq 3$, $\binom{n}{k_1, k_2, \cdots, k_m}$ is the number of $m-$ary vectors of length n over $[m]$ such that i occurs k_i times, which is also the number of ordered partitions of $[n]$ into m parts such that the i_{th} part has size k_i.

The following theorem is a generalization of the binomial theorem.

Exercise 1.26 (Multinomial Theorem). *For any reals $x_1, \ldots, x_m$ and any positive integer $n \geq 1$, we have*

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n,\ k_j \geq 0} \binom{n}{k_1, k_2, \cdots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}.$$

Exercise 1.27. *Suppose $\sum_{i=1}^{m} k_i = n$ with $k_i \geq 1$ for all $i \in [m]$. Then*

$$\binom{n}{k_1, k_2, \cdots, k_m} = \binom{n-1}{k_1 - 1, k_2, \cdots, k_m} + \cdots + \binom{n-1}{k_1, k_2, \cdots, k_m - 1}.$$

1.4 Inclusion and Exclusion Principle (IEP)

This lecture is devoted to Inclusion-Exclusion formula and its applications.

Let Ω be a ground set and let $A_1, A_2, ..., A_n$ be subsets of Ω. Write $A_i^c = \Omega \backslash A_i$. Throughout this lecture, we use the following notation.

Definition 1.28. *Let $A_\emptyset = \Omega$. For any nonempty subset $I \subseteq [n]$, let*

$$A_I = \bigcap_{i \in I} A_i.$$

For any integer $k \geq 0$, let

$$S_k = \sum_{I \in \binom{[n]}{k}} |A_I|.$$

Now we introduce Inclusion-Exclusion formula (in three equivalent forms) and give two proofs as follows.

Theorem 1.29 (Inclusion-Exclusion Formula)**.** *We have*

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} S_k,$$

which is equivalent to

$$\left| \Omega \backslash \bigcup_{i=1}^{n} A_i \right| = |A_1^c \cap A_2^c \cap \cdots \cap A_n^c| = \sum_{k=0}^{n} (-1)^k S_k,$$

and

$$\left| \Omega \backslash \bigcup_{i=1}^{n} A_i \right| = |A_1^c \cap A_2^c \cap \cdots \cap A_n^c| = \sum_{I \subseteq [n]} (-1)^{|I|} |A_I|.$$

Proof (1). For any subset $X \subseteq \Omega$, we define its characterization function $\mathbb{1}_X : \Omega \to \{0, 1\}$ by assigning

$$\mathbb{1}_X(x) = \begin{cases} 1, & x \in X \\ 0, & x \notin X. \end{cases}$$

Then we notice that $\sum_{x \in \Omega} \mathbb{1}_X(x) = |X|$. Let $A = A_1 \cup A_2 \cup \cdots \cup A_n$. Our key observation is that

$$(\mathbb{1}_A - \mathbb{1}_{A_1})(\mathbb{1}_A - \mathbb{1}_{A_2}) \cdots (\mathbb{1}_A - \mathbb{1}_{A_n})(x) \equiv 0,$$

which holds for any $x \in \Omega$. Next we expand this product into a summation of 2^n terms as follows:

$$\mathbb{1}_A + \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|} (\prod_{i \in I} \mathbb{1}_{A_i}) \equiv 0$$

holds for any $x \in \Omega$. Summing over all $x \in \Omega$, this gives that

$$|A| + \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|} |A_I| = 0,$$

which implies that

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} |A_I| = \sum_{k=1}^{n} (-1)^{k+1} S_k,$$

finishing the proof. ∎

Proof (2). It suffices to prove that

$$\mathbb{1}_{A_1 \cup A_2 \cup \cdots \cup A_n}(x) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{I \in \binom{[n]}{k}} \mathbb{1}_{A_I}(x)$$

holds for all $x \in \Omega$. Denote by LHS (resp. RHS) the left-hand side (resp. right-hand side) of the above equation.

Assume that x is contained in exactly ℓ subsets, say $A_1, A_2, \ldots, A_\ell$. If $\ell = 0$, then clearly $LHS = 0 = RHS$, so we are done. So we may assume that $\ell \geq 1$. In this case, we have $LHS = 1$ and

$$RHS = \ell - \binom{\ell}{2} + \binom{\ell}{3} + \cdots + (-1)^{\ell+1} \binom{\ell}{\ell} = 1.$$

Note that the above equation holds since $\sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} = (1-1)^\ell = 0$. This finishes the proof. ∎

Next, we will demonstrate the power of Inclusion-Exclusion formula by using it to solve several problems.

Definition 1.30. *Let $\varphi(n)$ be the number of integers $m \in [n]$ which are relatively prime*[1] *to n.*

Theorem 1.31. *If we express $n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$, where $p_1, \ldots, p_t$ are distinct primes, then*

$$\varphi(n) = n \prod_{i=1}^{t} (1 - \frac{1}{p_i}).$$

Proof. Let the ground set

$$\Omega = [n]$$

and

$$A_i = \{m \in [n] : p_i | m\}$$

for $i \in \{1, 2, \ldots, t\}$. It implies

$$\varphi(n) = \big|\{m \in [n] : m \notin A_i \text{ for all } i \in [t]\}\big| = \big|[n] \backslash (A_1 \cup A_2 \cup \cdots \cup A_t)\big|.$$

By Inclusion-Exclusion formula,

$$\varphi(n) = \sum_{I \subseteq [t]} (-1)^{|I|} |A_I|,$$

[1]Here, "m is relatively prime to n" means that the greatest common divisor of m and n is 1.

where $A_I = \cap_{i \in I} A_i = \{m \in [n] : (\prod_{i \in I} p_i) | m\}$ and thus $|A_I| = \frac{n}{\prod_{i \in I} p_i}$. We can derive that

$$\varphi(n) = \sum_{I \subseteq [t]} (-1)^{|I|} \frac{n}{\prod_{i \in I} p_i} = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots (1 - \frac{1}{p_t}),$$

as desired.

∎

Exercise 1.32. *For any positive integer n,*

$$\sum_{d|n} \varphi(d) = n$$

1.5 Möbius Inversion Formula

Definition 1.33. *The* **Möbius Function** μ *for a positive integer d is*

$$\mu(d) = \begin{cases} 1, & d \text{ is a product of even number of distinct primes } (d = 1 \text{ included}) \\ -1, & d \text{ is a product of odd number of distinct primes} \\ 0, & \text{otherwise} \end{cases}$$

Theorem 1.34. *For any positive integer n,*

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

Proof. If $n = 1$, it is trivial. For $n = p_1^{a_1} \dots p_r^{a_r} \geq 2$,

$$\sum_{d|n} \mu(d) = \sum_{i_1 \leq a_1, \dots, i_r \leq a_r} \mu(p_1^{i_1} \dots p_r^{i_r}) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i = 0.$$

∎

Theorem 1.35 (**Möbius Inversion Formula**)**.** *Let $f(n)$ and $g(n)$ be two functions defined for every positive integer n satisfying*

$$f(n) = \sum_{d|n} g(d).$$

Then we have

$$g(n) = \sum_{d|n} \mu(d) f(\frac{n}{d}).$$

Proof.

$$\begin{aligned}
\sum_{d|n} \mu(d) f(\frac{n}{d}) &= \sum_{d|n} \mu(\frac{n}{d}) f(d) \\
&= \sum_{d|n} \mu(\frac{n}{d}) (\sum_{d'|d} g(d')) \\
&= \sum_{d'|n} g(d') \sum_{\frac{n}{d} | \frac{n}{d'}} \mu(\frac{n}{d}) \\
&= \sum_{d'|n} g(d') \sum_{m | \frac{n}{d'}} \mu(m) \\
&= \sum_{d'|n, d' \neq n} g(d') \times 0 + g(n) \times 1 \\
&= g(n)
\end{aligned}$$

as desired. ∎

1.6 Generating Functions

Definition 1.36. *The (ordinary)* **generating function (GF)** *for an infinite sequence* $\{a_0, a_1, \dots\}$ *is a power series*

$$f(x) = \sum_{n \geq 0} a_n x^n.$$

We have two ways to view this power series.

(i). When the power series $\sum_{n \geq 0} a_n x^n$ converges (i.e. there exists a radius $R > 0$ of convergence), we view GF as a function of x and we can apply operations of calculus on it (including derivation and integration). For example, we know that

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

Recall the following sufficient condition on the radius of convergence that if $|a_n| \leq K^n$ for some $K > 0$, then $\sum_{n \geq 0} a_n x^n$ converges in the interval $(-\frac{1}{K}, \frac{1}{K})$.

(ii). When we are not sure of the convergence, we view the generating function as a formal series and take additions and multiplications. Let $a(x) = \sum_{n \geq 0} a_n x^n$ and $b(x) = \sum_{n \geq 0} b_n x^n$.

Addition.

$$a(x) + b(x) = \sum_{n \geq 0} (a_n + b_n) x^n.$$

Multiplication. Let $c_n = \sum_{i=0}^{n} a_i b_{n-i}$. Then

$$a(x) \cdot b(x) = \sum_{n \geq 0} c_n x^n.$$

Example 1.37. *Consider the GF of* $\{1, 1, 1, \dots\}$*. We note* $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ *holds for all* $-1 < x < 1$*. From the point view of (i), its first derivative gives*

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n.$$

So we could view $\frac{1}{(1-x)^2}$ *as the GF of* $\{1, 2, 3, \dots\}$ *for all* $-1 < x < 1$*.*

Problem 1.38. *Let* $a_0 = 1$ *and* $a_n = 2a_{n-1}$ *for* $n \geq 1$*. Find* a_n*.*

Solution. Consider the generating function,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} a_n x^n = 1 + 2x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = 1 + 2xf(x).$$

So $f(x) = \frac{1}{1-2x}$, which implies that $f(x) = \sum_{n=0}^{+\infty} 2^n x^n$ and $a_n = 2^n$. ∎

From this problem, we see one of the basic ideas for using generating function: in order to find the general expression of a_n, we work on its generating function $f(x)$; once we find the formula of $f(x)$, then we can expand $f(x)$ into a power series and get a_n by choosing the coefficient of the right term.

Problem 1.39. *Let* A_n *be the set of strings of length* n *with entries from the set* $\{a, b, c\}$ *and with no "aa" occuring (in the consecutive positions). Find* $|A_n|$ *for* $n \geq 1$*.*

Solution. Let $a_n = |A_n|$. We first observe that $a_1 = 3, a_2 = 8$. For $n \geq 3$, we will find a_n by recursion as follows. If the first string is 'a', the second string has two choices, 'b' or 'c'. Then the last $n-2$ strings have a_{n-2} choices. If the first string is 'b' or 'c', the last $n-1$ strings have a_{n-1} choices. They are all different. Totally, for $n \geq 3$, we have

$$a_n = 2a_{n-1} + 2a_{n-2}.$$

Set $a_0 = 1$, then $a_n = 2a_{n-1} + 2a_{n-2}$ holds for $n \geq 2$. The generating function of $\{a_n\}$ is

$$f(x) = \sum_{n\geq 0} a_n x^n = a_0 + a_1 x + \sum_{n \geq 2} (2a_{n-1} + 2a_{n-2})x^n = 1 + 3x + 2x(f(x) - 1) + 2x^2 f(x),$$

which implies that

$$f(x) = \frac{1+x}{1 - 2x - 2x^2}.$$

By Partial Fraction Decomposition, we calculate that

$$f(x) = \frac{1-\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}+1+2x} + \frac{1+\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}-1-2x},$$

which implies that

$$a_n = \frac{1-\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}+1} \left(\frac{-2}{\sqrt{3}+1}\right)^n + \frac{1+\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}-1} \left(\frac{2}{\sqrt{3}-1}\right)^n.$$

∎

Remark 1.40. *Note that a_n must be an integer but its expression is a combination of irrational terms! Observe that $\left|\frac{-2}{\sqrt{3}+1}\right| < 1$, so $\left(\frac{-2}{\sqrt{3}+1}\right)^n \to 0$ as $n \to \infty$. Thus, when n is sufficiently large, this integer a_n is about the value of the second term $\frac{1+\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}-1} \left(\frac{2}{\sqrt{3}-1}\right)^n$. Equivalently a_n will be the nearest integer to that.*

Exercise 1.41. *Define Fibonacci number F_n as follows: $F_1 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Find F_n.*

Definition 1.42. *For any real r and an integer $k \geq 0$, let*

$$\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}.$$

Exercise 1.43. *Prove that $\binom{\frac{1}{2}}{k} = \frac{(-1)^{k-1}\cdot 2}{4^k} \frac{(2k-2)!}{k!(k-1)!}$.*

Theorem 1.44 (Newton's Binomial Theorem)**.** *For any real number r and $x \in (-1, 1)$,*

$$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k.$$

Proof. By Taylor series, it is obvious. ∎

Corollary 1.45. *Let $r = -n$ for some integer $n \geq 0$. Then*

$$\binom{-n}{k} = \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!} = (-1)^k \binom{n+k-1}{k}.$$

Therefore

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k,$$

which is equivalent to

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k \tag{1}$$

Noting that

$$\binom{n+k-1}{k} = \#\text{ integer solutions to } x_1 + x_2 + \cdots + x_n = k \text{ where } x_i \geq 0, 1 \leq i \leq n,$$

we can explain Equation (1) from another point of view as follows.

Recall the following facts.

Fact 1.46. *For $j \in [n]$, let $f_j(x) := \sum_{i \in I_j} x^i$, where $I_j \subset \mathbb{N}$. Let b_k be the number of solutions to $i_1 + i_2 + \cdots + i_n = k$ for $i_j \in I_j$. Then*

$$\prod_{j=1}^{n} f_j(x) = \sum_{k=0}^{\infty} b_k x^k.$$

Fact 1.47. *If $f(x) = \prod_{i=1}^{k} f_i(x)$ for polynomials $f_1, ..., f_k$, then*

$$[x^n]f = \sum_{i_1+i_2+\cdots+i_k=n} \prod_{j=1}^{k} \left([x^{i_j}]f_j\right),$$

where $[x^n]f$ is the coefficient of x^n in f.

Let $f_j = (1-x)^{-1} = \sum_{i\geq 0} x^i$, $\forall j \in [n]$. By Fact 1.46, we can get Equation 1 by considering as $(1-x)^{-n} = \prod_{j=1}^{n} f_j$ easily.

Exercise 1.48. *Show $(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$ by taking the n^{th} derivative of $(1-x)^{-1}$.*

Problem 1.49. *Let a_n be the number of ways to pay n Yuan using 1-Yuan bills, 2-Yuan bills and 5-Yuan bills. What is the generating function of this sequence $\{a_n\}$?*

Solution. Observe that a_n is the number of integer solutions (i_1, i_2, i_3) to $i_1 + i_2 + i_3 = n$, where $i_1 \in I_1 := \{0, 1, 2, ...\}$, $i_2 \in I_2 := \{0, 2, 4, ...\}$ and $i_3 \in I_3 := \{0, 5, 10, ...\}$. Let $f_j(x) := \sum_{m\in I_j} x^m$ for $j = 1, 2, 3$. By Fact 1.46, we have

$$\sum_{n=0}^{+\infty} a_n x^n = f_1(x)f_2(x)f_3(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^5}.$$

∎

1.7 Random Walks

Consider a real axis with integer points $(0, \pm 1, \pm 2, \pm 3, \ldots)$ marked. A frog leaps among the integer points according to the following rules:

(1). At beginning, it sits at 1.

(2). In each coming step, the frog leaps either by distance 2 to the right (from i to $i+2$), or by distance 1 to the left (from i to $i-1$), each of which is randomly chosen with probability $\frac{1}{2}$ independently of each other.

Problem 1.50. *What is the probability that the frog can reach "0"?*

Solution. In each step, we use "+" or "−" to indicate the choice of the frog that is either to leap right or leap left. Then the probability space Ω can be viewed as the set of infinite vectors, where each entry is in $\{+, -\}$.

Let A be the event that the frog reaches "0". Let A_i be the event that the frog reaches "0" at the i^{th} step for the first time. So $A = \cup_{i=1}^{+\infty} A_i$ is a disjoint union. So $P(A) = \sum_{i=1}^{+\infty} P(A_i)$.

To compute $P(A_i)$, we can define a_i to be the number of trajectories (or vectors) of the first i steps such that the frog starts at "1" and reaches "0" at the i^{th} step for the first time. So

$$P(A_i) = \frac{a_i}{2^i}.$$

Then,

$$P(A) = \sum_{i=1}^{+\infty} \frac{a_i}{2^i}.$$

Let $f(x) = \sum_{i=0}^{+\infty} a_i x^i$ be the generating function of $\{a_i\}_{i \geq 0}$, where $a_0 := 0$. Thus,

$$P(A) = \sum_{i=1}^{+\infty} \frac{a_i}{2^i} = f\left(\frac{1}{2}\right).$$

We then turn to find the expression of $f(x)$.

Let b_i be the number of trajectories of the first i steps such that the frog starts at "2" and reaches "0" at the i^{th} step for the first time.

Let c_i be the number of trajectories of the first i steps such that the frog starts at "3" and reaches "0" at the i^{th} step for the first time.

First we express b_i in terms of $\{a_j\}_{j \geq 1}$. Since the frog only can leap to left by distance 1, if the frog can successfully jump from "i" to "0" in i steps, then this frog must reach "1" first. Let j be the number of steps by which the frog reaches "1" for the first time. So there are a_j trajectories from "2" to "1" at the j^{th} step for the first time. In the remaining $i-j$ steps the frog must jump from "1" to "0" and reach "0" at the coming $(i-j)^{th}$ step for the first time, so there are a_{i-j} trajectories that the frog can finish in exactly $i-j$ steps. In total,

$$b_i = \sum_{j=1}^{i-1} a_j a_{i-j}.$$

As $a_0 = 0$,

$$b_i = \sum_{j=0}^{i} a_j a_{i-j}.$$

We can get

$$\sum_{i \geq 0} b_i x^i = (\sum_{i \geq 0} a_i x^i)^2 = f^2(x).$$

Similarly, if we count the number c_i of trajectories from 3 to 0, we can obtain that

$$c_i = \sum_{j=0}^{i} a_j b_{i-j},$$

which implies that

$$\sum_{i \geq 0} c_i x^i = \left(\sum_{i \geq 0} b_i x^i\right)\left(\sum_{i \geq 0} a_i x^i\right) = f^3(x).$$

Let us consider a_i from another point of view. After the first step, either the frog reaches "0" directly (if it leaps to left, so $a_1 = 1$), or it leaps to "3". In the latter case, the frog needs to jump from "3" to "0" using $i-1$ steps. Thus for $i \geq 2$, $a_i = c_{i-1}$.

Combining the above facts, we have

$$f(x) = \sum_{i=0}^{+\infty} a_i x^i = x + \sum_{i \geq 2} a_i x^i = x + \sum_{i \geq 2} c_{i-1} x^i = x + x\left(\sum_{j=0}^{+\infty} c_j x^j\right) = x + x \cdot f^3(x).$$

Let $a := P(A) = f(1/2)$. Then we have $a = \frac{1}{2} + \frac{a^3}{2}$, i.e., $(a-1)(a^2+a-1) = 0$, implying that

$$a = 1,\ \frac{\sqrt{5}-1}{2} \text{ or } \frac{-\sqrt{5}-1}{1}.$$

Since $P(A) \in [0,1]$, we see $P(A) = 1$ or $\frac{\sqrt{5}-1}{2}$.

Note that $f(x) = x + xf^3(x)$. Consider the inverse function of $f(x)$, that is, $g(x) := \frac{x}{1+x^3}$. Consider the figure of $g(x)$. We find that $g(x)$ is increasing around $\frac{\sqrt{5}-1}{2}$ but decreasing around 1. Since $f(x) = \sum a_i x^i$ is increasing, $g(x)$ also increases. Thus it doesn't make sense for $g(x)$ being around $x = 1$. This explains that $P(A) = \frac{\sqrt{5}-1}{2}$, which is the golden section! ∎

1.8 Exponential Generating Functions

Let $\mathbb{N}, \mathbb{N}_e$ and $\mathbb{N}_o$ be the sets of non-negative integers, non-negative even integers and non-negative odd integers, respectively.

Given n sets I_j of non-negative integers for $j \in [n]$, let $f_j(x) = \sum_{i \in I_j} x^i$. Let a_k be the number of integer solutions to $i_1 + i_2 + \cdots + i_n = k$, where $i_j \in I_j$. Then $\prod_{j=1}^n f_j(x)$ is the ordinary generating function of $\{a_k\}_{k \geq 0}$.

Problem 1.51. *Let S_n be the number of selections of n letters chosen from an unlimited supply of a's, b's and c's such that both of the numbers of a's and b's are even.*

Solution. We can write S_n as

$$S_n = \sum_{e_1+e_2+e_3=n,\ e_1,e_2 \in \mathbb{N}_e,\ e_3 \in \mathbb{N}} 1.$$

Using the previous fact, we see that $S_n = [x^n]f$, where

$$f(x) = \left(\sum_{i \in \mathbb{N}_e} x^i\right)^2 \left(\sum_{j \in \mathbb{N}} x^j\right) = \left(\frac{1}{1-x^2}\right)^2 \cdot \frac{1}{1-x}.$$

∎

Problem 1.52. *Let T_n be the number of arrangements (or words) of n letters chosen from an unlimited supply of a's, b's and c's such that both of the numbers of a's and b's are even. What is the value of T_n?*

Solution. To solve this, we define a new kind of generating functions.

Definition 1.53. *The exponential generating function for the sequence $\{a_n\}_{n \geq 0}$ is the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot \frac{x^n}{n!}.$$

Then we have the following fact.

Fact 1.54. *If we have n letters including x a's, y b's and z c's (i.e. $x + y + z = n$), then we can form $\frac{n!}{x!y!z!}$ distinct words using them.*

Therefore, a selection (say x a's, y b's and z c's) can contribute $\frac{n!}{x!y!z!}$ arrangements to T_n. This implies that

$$T_n = \sum_{e_1+e_2+e_3=n,\ e_1,e_2\in\mathbb{N}_e,\ e_3\in\mathbb{N}} \frac{n!}{e_1!e_2!e_3!}.$$

Similar to defining the above $f(x)$ for S_n, we define the following for T_n. Let

$$g(x) := \left(\sum_{i\in\mathbb{N}_e} \frac{x^i}{i!}\right)^2 \left(\sum_{j\in\mathbb{N}} \frac{x^j}{j!}\right).$$

Claim. We have

$$[x^n]g = \frac{T_n}{n!}.$$

Proof. To see this, we expand $g(x)$. Then the term x^n in $g(x)$ becomes

$$\sum_{\substack{e_1+e_2+e_3=n,\\ e_1,e_2\in\mathbb{N}_e,\ e_3\in\mathbb{N}}} \frac{x^{e_1}}{e_1!}\cdot\frac{x^{e_2}}{e_2!}\cdot\frac{x^{e_3}}{e_3!} = \left(\sum_{\substack{e_1+e_2+e_3=n,\\ e_1,e_2\in\mathbb{N}_e,\ e_3\in\mathbb{N}}} \frac{n!}{e_1!e_2!e_3!}\right)\frac{x^n}{n!} = T_n\cdot\frac{x^n}{n!}.$$

So $[x^n]g = \frac{T_n}{n!}$, i.e., $g(x)$ is the exponential generating function of $\{T_n\}$. This finishes the proof of Claim. ∎

Using Taylor series: $e^x = \sum_{j\geq 0}\frac{x^j}{j!}$ and $e^{-x} = \sum_{j\geq 0}(-1)^j\frac{x^j}{j!}$, we have

$$\frac{e^x+e^{-x}}{2} = \sum_{j\in\mathbb{N}_e}\frac{x^j}{j!} \quad\text{and}\quad \frac{e^x-e^{-x}}{2} = \sum_{j\in\mathbb{N}_o}\frac{x^j}{j!}.$$

By the previous fact, we get

$$g(x) = \left(\frac{e^x+e^{-x}}{2}\right)^2\cdot e^x = \frac{e^{3x}+2e^x+e^{-x}}{4} = \sum_{n\geq 0}\left(\frac{3^n+2+(-1)^n}{4}\right)\cdot\frac{x^n}{n!}.$$

Therefore, we get that

$$T_n = \frac{3^n+2+(-1)^n}{4}.$$

∎

Recall that the *exponential generating function* for the sequence $\{a_n\}_{n\geq 0}$ is the power series

$$f(x) = \sum_{n=0}^{+\infty} a_n\cdot\frac{x^n}{n!}.$$

As we shall see, ordinary generation functions can be used to find the number of selections; while exponential generation functions can be used to find the number of arrangements or some combinatorial objects **involving ordering**. We summarize this as the following facts.

Fact 1.55. *Given $I_j \subseteq \mathbb{N}$ for $j \in [n]$, let $f_j(x) = \sum_{i \in I_j} x^i$. And let $a_k = \sum_{\substack{i_1+\cdots+i_n=k,\\ i_j \in I_j}} 1$. Then*

$$\prod_{j=1}^{n} f_j(x) = \sum_{k=0}^{+\infty} a_k x^k.$$

Fact 1.56. *Given $I_j \subseteq \mathbb{N}$ for $j \in [n]$, let $g_j(x) = \sum_{i \in I_j} \frac{x^i}{i!}$. And let $b_k = \sum_{\substack{i_1+\cdots+i_n=k,\\ i_j \in I_j}} \frac{k!}{i_1!i_2!\cdots i_n!}$. Then*

$$\prod_{j=1}^{n} g_j(x) = \sum_{k=0}^{+\infty} \frac{b_k}{k!} x^k.$$

Fact 1.57. *Let $f(x) = \prod_{j=1}^{n} f_j(x)$. Then*

$$[x^k]f = \sum_{\substack{i_1+\cdots+i_n=k,\\ i_j \geq 0}} \prod_{j=1}^{n} [x^{i_j}] f_j.$$

Fact 1.58. *Let $f(x) = \prod_{j=1}^{n} f_j(x)$ and let $f_j(x) = \sum_{k=0}^{+\infty} \frac{a_k^{(j)}}{k!} x^k$. Then*

$$f(x) = \sum_{k=0}^{+\infty} \frac{A_k}{k!} x^k,$$

if and only if

$$A_k = \sum_{\substack{i_1+\ldots+i_n=k,\\ i_j \geq 0}} \frac{k!}{i_1!i_2!\cdots i_n!} \Big(\prod_{j=1}^{n} a_{i_j}^{(j)} \Big).$$

Exercise 1.59. *Find the number a_n of ways to send n students to four different classes (say R_1, R_2, R_3, R_4) such that each class has at least one student.*

Solution.

$$a_n = \sum_{\substack{i_1+i_2+i_3+i_4=n,\\ i_j \geq 1}} \frac{n!}{i_1!i_2!i_3!i_4!}.$$

Let $I_j \subseteq \mathbb{N}$ for $j \in [4]$ and $g_j(x) = \sum_{i \geq 1} \frac{x^i}{i!} = e^x - 1$. By Fact 1.56, we have that

$$\sum_{n=0}^{+\infty} \frac{a_n}{n!} x^n = g_1 g_2 g_3 g_4 = (\sum_{i \geq 1} \frac{x^i}{i!})^4 = (e^x - 1)^4 = e^{4x} - 4e^{3x} + 6e^{2x} - 4e^x + 1 = \sum_{n=0}^{+\infty} (4^n - 4\cdot 3^n + 6\cdot 2^n - 4) \frac{x_n^n}{n!} + 1.$$

Thus $a_n = 4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4$ for $n \geq 4$. ∎

Exercise 1.60. *Let a_n be the number of arrangements of type A for a group of n people, and let b_n be the number of arrangements of type B for a group of n people.*

Define a new arrangement of n people called type C as follows:

- *Divide the n people into 2 groups (say 1^{st} and 2^{nd}).*
- *Then arrange the 1^{st} group by an arrangement of type A, and arrange the 2^{nd} group by an arrangement of type B.*

Let c_n be the number of arrangements of type C of n people. Let $A(x), B(x), C(x)$ be the exponential generation function for $\{a_n\}, \{b_n\}, \{c_n\}$ respectively. Prove that $C(x) = A(x)B(x)$.

Proof. We can easily see that

$$c_n = \sum_{\substack{i+j=n,\\ i,j\geq 0}} \frac{n!}{i!j!} a_i b_j.$$

Then by Fact 1.58, $C(x) = A(x)B(x)$. ∎

Exercise 1.61. *Recall that $S(n,k) \cdot k!$ is equal to the number of surjections from $[n]$ to $[k]$. For fixed k, compute the exponential generating function of $S(n,k) \cdot k!$. Then find the value of $S(n,k) \cdot k!$.*