Formalizing modular forms

RUC/YMSC Workshop on Formalized Mathematics

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General motivation

- **Modular forms**: central to much current research in number theory
- Very **wide pre-requisites**:
  - calculus
  - measure theory
  - group theory
  - differential geometry
  - (eventually) algebraic geometry, representation theory, …

$\Rightarrow$ good “stress-test” for **formal libraries**

- Early formalization efforts (Feit–Thompson, Liquid Tensor Experiment) focussed on quite **self-contained** areas – this is the opposite
- relation to **sphere packing** (more on this later)

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A personal motivation

- ***p*-adic *L*-functions** attached to modular forms – related to complex *L*-functions via “interpolation formula”
- Remark by **M. Fütterer**, 2015: two contradictory versions of the formula in the literature (roughly half-half split)
- Either Mazur–Tate–Teitelbaum or Fukaya–Kato is wrong, and **I don’t know which**!
- Literature full of many minor errors (and some more **serious** ones)

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Prior work

DL + Michael Stoll, 2024-5: Lean formalization of **Riemann zeta function** (& Dirichlet *L*-functions)

- Analytic continuation & functional equation (using Poisson summation and θ-function)
- $L(\chi, 1) \neq 0$, and hence Dirichlet's theorem on primes in AP's
- Precise **formal statement of Riemann Hypothesis**

Emphasis on *analytic* number theory – continued in **"PNT+"** (Kontorovich, Tao)

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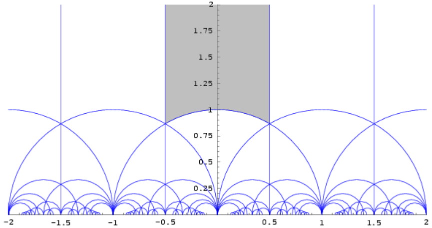
Crash course in modular form theory

- $\mathcal{H}$ upper half-plane, $\mathrm{GL}_2^+(\mathbb{R}) \curvearrowright \mathcal{H}$ by Moebius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

- $\mathrm{SL}_2(\mathbb{Z})$ generated by $z \mapsto z+1$ and $z \mapsto \frac{-1}{z}$



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Modular forms

- Consider: $\Gamma \le \mathrm{SL}_2(\mathbb{Z})$ finite-index, $k \in \mathbb{Z}$
- **Definition**: A modular form is a function $f : \mathcal{H} \to \mathbb{C}$ which is:
  - **Holomorphic** on $\mathcal{H}$
  - Weight k Γ**-invariant**: $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$
  - **Bounded at cusps**: $f(z)$ has Fourier expansion $\sum_{\mathbf{n \geq 0}} a_n q^n$, $q = e^{2\pi i n z}$
    (and similar condition for all translates by $\mathrm{SL}_2 \mathbb{Z}$)
- $M_k(\Gamma)$ space of these, $S_k(\Gamma)$ subspace of *cusp forms* (i.e. $a_0 = 0$)
- Basic theorem: $M_k(\Gamma)$ is finite-dimensional $/\mathbb{C}$,
  and zero if $k < 0$

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Previous formalization

- Actions on $\mathcal{H}$, definition of $M_k(\Gamma)$: in Mathlib for many years

- More recently: C. Birkbeck formalized **Eisenstein series**

$$G_k(z) = \sum_{(c,d)\neq(0,0)} \frac{1}{(cz+d)^k} \in M_k(\mathrm{SL}_2\,\mathbb{Z}) \qquad k \geq 4$$

- Takes nontrivial work to show this is a modular form (locally uniform absolute convergence) and to compute q-expansion

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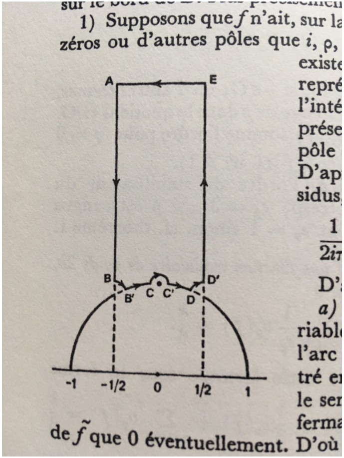
Finite-dimensionality

- Textbook proof (level $\mathrm{SL}_2\,\mathbb{Z}$): use *valence formula*

$$v_\infty(f) + \tfrac{1}{2} v_i(f) + \tfrac{1}{3} v_\rho(f) + \sum_{\substack{z \in \mathrm{SL}_2\,\mathbb{Z}\backslash\mathcal{H} \\ z \neq i,\rho}} v_z(f) = \frac{k}{12}.$$

- Proof uses integral of $\frac{f'}{f}$ around boundary of fundamental domain



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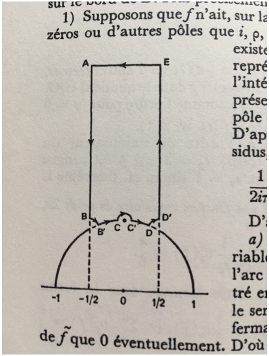
Finite-dimensionality

- Very **difficult to formalize**
- Tedious to write down explicit parametrisation of contour
- Need some version of Jordan curve theorem (which points are “inside” contour?)
- Gets much worse if *f* has zeroes on the boundary of fundamental domain
- 3 approaches:
  - “Detour calculus” (Eberl, in Isabelle)
  - Cauchy principal value & non-integer winding numbers
  - Or : **abandon contour integrals totally**



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Alternative proof of finite-dimensionality

- Step 1: show $\boxed{M_k(\mathrm{SL}_2\,\mathbb{Z}) = 0 \text{ for } k < 0}$, using **maximum modulus principle** applied to $\tilde{f}$, where $f(z) = \tilde{f}(q)$
  - Suggested by Alex Kontorovich
  - Formalized by Birkbeck + DL, autumn 2024
- Step 2: consider $\Delta(z) = q\prod_{n\geq 1}(1 - q^n)^{24}$
  - Straightforward to show this is holomorphic with a simple zero at ∞ and non-vanishing on $\mathcal{H}$
  - Much harder: show it transforms correctly under $\frac{-1}{z}$
  - Use Eisenstein series G_2 (almost, but not quite, modular in weight 2)
  - Show $\operatorname{dlog}\Delta = G_2$
- Thus multiplication by Δ gives isomorphism $M_k \to S_{k+12}\ \forall k \in \mathbb{Z}$, hence **finite-dimensionality**

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General levels

- Finite-dimensionality in **general levels**: many textbooks use heavy machinery of Riemann surfaces, sheaves, Riemann–Roch theorem
- Alternative argument (DL, 2010): use *norm maps*

$$f \in M_k(\Gamma) \mapsto \prod_{\gamma \in \Gamma \backslash \mathrm{SL}_2 \mathbb{Z}} (f \mid_k \gamma) \in M_{kd}(\mathrm{SL}_2 \mathbb{Z}), \quad d = [\mathrm{SL}_2 \mathbb{Z} : \Gamma]$$

  $\rightsquigarrow$ can deduce finite-dimensionality of $M_k(\Gamma)$ from that of $M_{kd}(\mathrm{SL}_2 \mathbb{Z})$ (WIP; only vanishing for $k < 0$ formalized so far)
- This method can't give exact dimension formulae, but will give an upper bound (error = genus of $X(\Gamma)$)

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The sphere-packing problem

- What is the densest packing of spheres in *n*-dimensional space?
- 3-dimensional case is **Kepler's conjecture** (formalized by Hales et al)
- **Viazovska's theorem**: Optimal sphere packing in 8-dimensional space given by E_8 lattice
- Much **cleaner** than 3-dim'l case but uses more technology; **modular forms** (and quasimodular forms) play major role

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Formalizing sphere packing

- Birkbeck, Hariharan, Lee: project to formalize E_8 optimality
- Needs exact **dimension formulae** for level 1 mod forms spaces, and estimates for **growth of q-expansions**
- Blueprint used my strategy above (developed in conversations with Chris Birkbeck)

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Enter Gauss

- Autumn 2025: Birkbeck et al invited AI teams to contribute to their project (public blueprint)
- Math, Inc.: **Gauss** autoformalization agent
- Trickle of small PR's then long silence...
- Feb 2026: complete proof of E_8 optimality (200,000 lines)
- Since slimmed down to 50,000 lines, code quality not brilliant (but not terrible either)
- Includes complete proof of modular form finite-dimensionality (all levels)

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Lessons learned

- “Local” parts of **complex analysis** formalize well, “global” geometric theory much harder
- Subtypes:

$$\mathsf{SL}_2(\mathbb{Z}) \subset \mathsf{SL}_2(\mathbb{Q}) \subset \mathsf{GL}_2^+(\mathbb{Q}) \subset \mathsf{GL}_2^+(\mathbb{R}) \subset \mathsf{GL}_2(\mathbb{R}) \ldots$$

$\rightsquigarrow$ **coercions** ad nauseam, “$\uparrow\uparrow\uparrow\uparrow g$”

- Extend $\mathsf{GL}_2^+(\mathbb{R})$ action to $\mathsf{GL}_2(\mathbb{R})$ (negative-determinant elements act conjugate-linearly on modular forms)
- Allow general discrete subgroups of $\mathsf{GL}_2(\mathbb{R})$

- Greater mathematical generality *also* avoids type-theoretic headaches

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Next steps

- Short term: develop **Hecke operators** and ***L*-functions of modular forms**
- P-adic *L*-functions (Fütterer's sign issue): might just be possible
- Angdinata, Birkbeck, DL: project to formalize **Heegner points** on *modular* elliptic curves

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