Last time, intersection number

$C \cdot D$, with C a curve
D a Cartier divisor.

Example: X smooth, D a prime smooth divisor
$C \subseteq X$ a smooth curve.
Assume that $C \cap D \neq \phi$, $C \not\subseteq D$.
Then $C \cdot D$ is just the number of intersection points of $C \cap D$ counted with multiplicity.

We can extend this intersection pair via linearity to $\mathbb{Q}$- Cartier divisors or even $\mathbb{R}$- Cartier divisors.

Cone of curves X proj normal

$Z_1(X) = \{\mathbb{Z}$- linear combinations of irr proj curves in $X\}$.

$Z_1(X)_{\mathbb{Q}} = Z_1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, $Z_1(X)_{\mathbb{R}} = Z_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$

Then $Z_1(X)_{\mathbb{R}} = \{\mathbb{R}$- linear combinations of irr proj curves $\}$.

Recall that $\mathrm{Pic}(X) = \{\text{Cartier divisors on } X\}$.

$$\mathrm{Pic}(X) = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad \mathrm{Pic}(X)_{\mathbb{R}} = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$$

$$\| \qquad\qquad\qquad \|$$

$$\{\mathbb{Q}\text{-Cartier divisors}\} \qquad \{\mathbb{R}\text{-Cartier divisors}\}$$

By definition, D is an $\mathbb{R}$-Cartier divisor if $D = a_1 D_1 + \cdots + a_k D_k$ such that a_i are real numbers, D_i are Cartier divisors (D_i not necessarily irreducible)

The intersection number extends to a pairing

$$Z_1(X)_{\mathbb{R}} \times \mathrm{Pic}(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$$

$$(C, D) \longmapsto C \cdot D$$

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by modding out the kernel of the pairing we can obtain a perfect pairing

$$N_1(X) \times N^1(X) \longrightarrow \mathbb{R}$$

$$[C], [D] \longmapsto C \cdot D$$

$N_1(X) = Z_1(X)_{\mathbb{R}} / -$, $N^1(X) = \mathrm{Pic}(X)_{\mathbb{R}} /$

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$\equiv$ means equivalent to zero for the intersection pairing.

That is, a divisor (class) D satisfies

$$D \equiv 0 \iff C \cdot D = 0 \text{ for all curves } C \in X.$$

"$\equiv$" is called "numerically equivalent"

Prop: $\dim_{\mathbb{R}} N^1(X) = \dim_{\mathbb{R}} N_1(X)$ is finite.

This number, denoted as $\rho(X)$ is called the Picard number of X.

Exercise: If $D \sim_{\mathbb{Q}} D'$ then $D \equiv D'$.

(KMM87 for definition of these spaces)
Section 0

Def: We define $\overline{NE}(X)$ as the closed convex cone in $N_1(X)$ generated by curve classes.

This is also called the Mori Cone of X.

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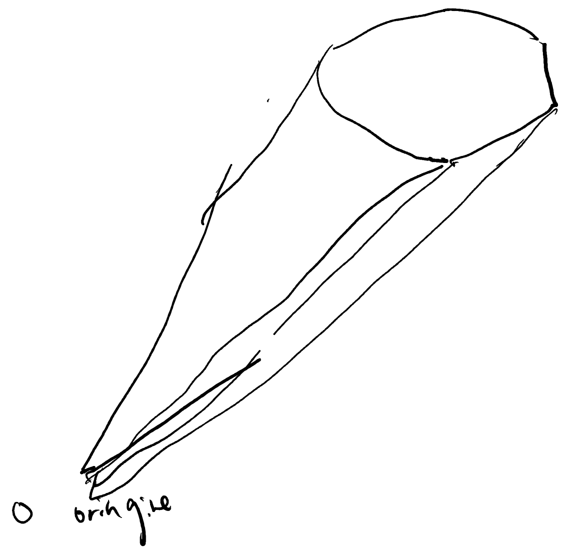
$$\overline{NE}(X) = \overline{\{a_1C_1 + \cdots + a_kC_k,\ a_i \in \mathbb{R}^+,\ C_i \text{ proj irr curve in } X\}}$$

$\subseteq N_1(X)$.

Note: $\{a_1C_1 + \cdots + a_kC_k,\ a_i \in \mathbb{R}^+,\ C_i \text{ proj irr curve}\}$

is always a convex cone, but not

closed ~~in~~ general.



Positive divisors

Theorem (Kleinman Criterion): X normal proj.

A Cartier D is ample iff $\delta \cdot D > 0$

for all class $\delta \in \overline{NE}(X) - \{0\}$

Recall that a divisor is ample if $\exists a \in \mathbb{N}_{>0}$ s.t.

aD is very ample, $\Longleftrightarrow$

if $(s_0, \cdots, s_N)$ is a basis of $H^0(X, \mathcal{O}_X(aD))$

then $\varphi: \begin{cases} X \longrightarrow \mathbb{P}^N \\ x \longmapsto [s_0(x) : \cdots : s_N(x)] \end{cases}$

is an embedding.

and $\varphi^* \mathcal{O}_{\mathbb{P}^N}(1) \simeq \mathcal{O}_X(aD)$.

(See Hartshorne for ample, very ample divisors).

Cor: If $D \equiv D'$, then D is ample $\Longleftrightarrow$ D' is ample

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Def: a $\mathbb{R}$-Cartier D divisor is ample if $\delta \cdot D > 0$

for $\delta \in \overline{NE}(X) - \{0\}$.

Def: an $\mathbb{R}$-Cartier divisor D is called NEF

if $\delta \cdot D \geqslant 0$ for $\delta \in \overline{NE}(X)$

Remark: D is nef $\Longleftrightarrow$ $C \cdot D \geq 0$ for any curve C

But this does not characterize ampleness

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Notation: $\mathrm{Amp}(X) \subseteq N^1(X)$ is the cone of ample divisors

$\mathrm{Nef}(X) \subseteq N^1(X)$ is the cone of nef divisors

Prop: $\overline{\mathrm{Amp}(X)} = \mathrm{Nef}(X)$, and $\mathring{\mathrm{Nef}}(X) = \mathrm{Amp}(X)$.

Def: A Cartier divisor D is called big if

$$\limsup_{m\to\infty} \frac{\dim H^0(X, \mathcal{O}_X(mD))}{m^n} > 0, \text{ where } n = \dim X$$

Prop: D is big $\Longleftrightarrow$ D can be written as

$$D \sim_{\mathbb{Q}} A + E$$

where A is an ample $\mathbb{Q}$-divisor,

E is an effective $\mathbb{Q}$-divisor (coeff ≥ 0)

Def: An $\mathbb{R}$-Cartier divisor D is big if

$D \sim_{\mathbb{R}} A + E$ with A ample, E eff

Def: We denote by $\mathrm{Psef}(X)$ the closure

$\{ a_1 D_1 + \cdots + a_k D_k, \quad a_i \geq 0, D_i$ prime divisor $y \subseteq N(X)$

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Psef is the cone of pseudo-effective divisors

Prop: $\mathrm{Big}(X)$ is the cone of big divisors.

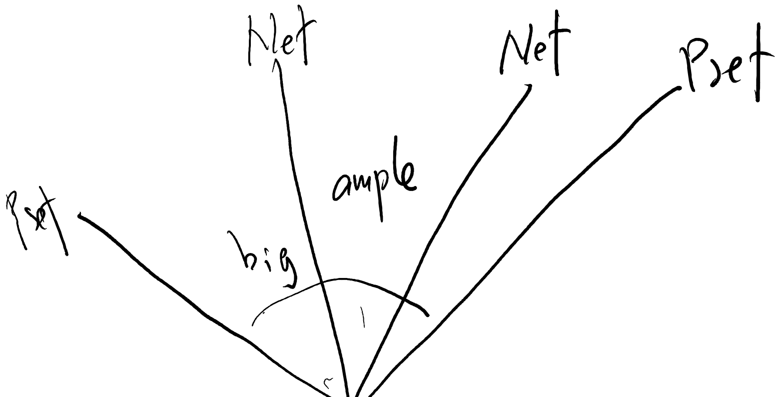
Then $\overline{\mathrm{Big}(X)} = \mathrm{Psef}(X)$

and $\mathring{\mathrm{Psef}}(X) = \mathrm{Big}(X)$.

Moreover, $\mathrm{Amp}(X) \subseteq \mathrm{Big}(X)$

$\mathrm{Nef}(X) \subseteq \mathrm{Psef}(X)$.

[KM98, Section 2.5]



$\subseteq N^1(X)$.

Idea of MMP: Classify X by looking at the numerical behavior of K_X.

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II Low dimensional MMP.

II.1 $\dim X = 1$.

If $\deg K_X < 0$ then $X \simeq \mathbb{P}^1$

If $\deg K_X = 0$ then X elliptic curve

If $\deg K_X > 0$, then X is a higher genus curve.

II.2 $\dim X = 2$.

Theorem (Castelnuovo Thm, see Hartshorne)

X a smooth surface. $C \subseteq X$ a smooth proj curve. Assume that $C \cdot C = -1$ and that $C \simeq \mathbb{P}^1$

(Such a C is called a (-1)-curve)

Then there is a birational morphism

$f: X \longrightarrow Y$ such that

① Y is smooth

② f is an isomorphism outside $C \subseteq X$

③ $f(C)$ is a point

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Prop: $X \xrightarrow[\text{(-1)-curve } C]{\text{blow down of}} Y$ to a point $y \in Y$

$X \dashleftarrow Y$: blowup of $y \in Y$.

MMP for surfaces — Given smooth proj surface X.

We can apply Castelnuovo Thm successively

$$X = X_0 \longrightarrow X_1 \longrightarrow X_2 \cdots \longrightarrow X_k = Z$$

such that X_k contains no (-1)-curve

This is the MMP for smooth proj surfaces.

Classification of $X_k = Z$. (Z is minimal in the sense that Z has no more Castelnuovo contraction)

Enriques Classification (~1900)

1. $-K_Z$ ample, then $Z \simeq \mathbb{P}^2$
2. Z is a ruled surface, Z is a $\mathbb{P}^1$-bundle over a curve C, $Z \longrightarrow C$
3. $K_Z \equiv 0$, there is a list
4. elliptic surfaces, $Z \longrightarrow C$ a fibration, general fibers are elliptic curves
5. K_Z ample, Z is called general type

It is difficult to generalize to Higher dim.

Particularly, it is indispensible to study sing var.

III MMP singularities (KM98, 2.3)

Let X be a normal variety.

Then its singular locus X_{sing} Satisfies $\operatorname{codim} X_{sing} \geq 2$

Hironaka Theorem: $\exists$ birational (proj surj) morphism $r: \tilde{X} \to X$ such that $\tilde{X}$ is smooth. Moreover, we can require that $r^{-1}(X_{sing})$ is a divisor with simple normal crossing support.

Def: A (reduced) divisor $D = D_1 + \cdots + D_k \subseteq X$ is called simple normal crossing if X is smooth along $\bigcup_{i=1}^{k} D_i$

and for any subset $J \subseteq \{1, \dots, k\}$.

$\bigcap_{i \in J} D_i$ is smooth (with multiplicity one)

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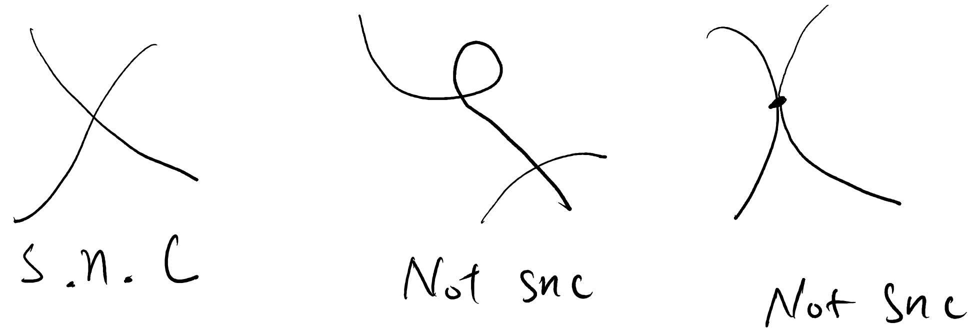
Example:



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in: $X = \mathbb{C}^n$, let $H_1, \cdots, H_n$ be the coordinate hyperplanes,

Then $D = H_1 + \cdots + H_n$ is snc

Simple fact, $D = D_1 + \cdots + D_k$ snc.

If $J \subseteq \{1, \cdots, k\}$, the $D' = \sum_{i \in J} D_i$ is again snc.

- Hironaka Thm means you can "replace" your X_{sing} by sing normal crossing divisor.

III.1 discrepency

Assume that K_X is $\mathbb{Q}$-Cartier.

Let $r: \tilde{X} \to X$ be a good resolution of singularities. Then we can pullback K_X that is $r^* K_X$

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We can compare it with $K_{\tilde{X}}$.

We can write
$$K_{\tilde{X}} \equiv r^* K_X + \sum a_i E_i$$

Since $\tilde{X}$ and X are birational, the E_i are in the exceptional locus of r
(r is not an iso around E_i)

Such E_i is called an exceptional divisor.

Def: a_i is called the discrepancy of E_i for X.

$a_i = a(E, X, 0)$.

Note: $a(E, X, 0)$ is independent of the choice of $\tilde{X}$, but the generic point of E.

If
$$\begin{tikzcd} W \arrow[r] \arrow[rr, bend right, "q"'] & \tilde{X} \arrow[r, "r"] & X \end{tikzcd}, \quad q : W \to X \text{ another resolution.}$$

Write $K_W \equiv q^* K_X + \sum b_j F_j$

If, up to renumbering, the strict transform of E_i in W is F_i

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Then $b_1 = a_1$.

Valuation viewpoint. A prime divisor $E \subseteq \tilde{X}$, $\tilde{X}$ birational to X can be viewed as a valuation ν_E on the field

$$K(X) = K(\tilde{X})$$

such that for $g \in K(X) = K(\tilde{X})$,

$$\nu_E(g) = \exp(-\mathrm{coeff}_E(\mathrm{div}\, g))$$

KM98, Sect 2.3