# Modern Mathematics Lecture online

# Fourier Transform and Finite Analogues

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We first recall some standard Fourier analysis:

(Reference for this part:

J. Fourier, "Théorie analytique de la chaleur" (1822)

="Analytical theory of heat" )

For $f: \mathbb{R} \to \mathbb{C}$ we have the Fourier transform of $f$:

$$\hat{f}: \mathbb{R} \to \mathbb{C}, \quad \hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx.$$

We have the following formula:

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i x \xi} d\xi.$$

We have $\hat{\hat{f}}(x) = f(-x)$. $\qquad \hat{\hat{\hat{\hat{f}}}} = f$.

Therefore the map $f \mapsto \hat{f}$ has eigenvalues $\pm 1, \pm i$.

Hermite (1864) described its eigenvectors:

$$x \mapsto \text{const} \cdot e^{-\pi x^2} H_n(2x\sqrt{\pi}), \quad n = 0, 1, \cdots$$

$H_n$ are certain polynomials. $H_0 = 1$, $H_1 = x$, $H_2 = x^2 - 1$, $H_3 = x^3 - 3x, \cdots$

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We now suppose that $V$ is a vector space of dimension $N$ over a finite field $\mathbb{F}_q$, with a nondegenerate bilinear form

$$\langle -,- \rangle : V \times V \to \mathbb{F}_q$$

such that there exists $\varepsilon \in \{\pm 1\}$, $\langle x,y \rangle = \varepsilon \langle y,x \rangle$.

Now for any $f \in \mathbb{C}^V := \{f : V \to \mathbb{C}\}$, we may define its Fourier transform by

$$\hat{f}(\xi) = \frac{1}{\sqrt{|V|}} \sum_{x \in V} f(x) \psi(\langle x,\xi \rangle), \quad \hat{f} \in \mathbb{C}^V.$$

where $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ is a group homomorphism not equal to 1.

We now have $\hat{\hat{f}}(x) = f(-\varepsilon x)$. In particular if $\varepsilon = -1$ then $\hat{\hat{f}} = f$.

We now set $q = 2$ (so that $\varepsilon = \pm 1$ does not matter) and that $\langle -,- \rangle$ is symplectic, i.e. $\langle x,x \rangle = 0, \forall x$

We also write $\dim V = N = 2n$, and set $\psi(x) = (-1)^x$.

In this case the Fourier transformation is given by

$$\hat{f}(\xi) = 2^{-n} \sum_{x \in V} f(x) (-1)^{\langle x,\xi \rangle}$$

For any $X \subseteq V$, we set $b_X : V \to \mathbb{C}$ to be the characteristic function of $X$. Then we have $\hat{b}_X = \frac{|X|}{2^n} b_{X^\perp}$, where

$X^\perp := \{y \in V \mid \langle x,y \rangle = 0, \forall x \in X\}$. In particular, if $X = X^\perp$ (i.e. $X$ is a Lagrangian subspace), then $b_X = \hat{b}_X$.

We also notice that the dimension of the (+1)-eigenspace is $2^{2n-1}+2^{n-1}$ and the dimension of the $(-1)$-eigenspace is $2^{2n-1}-2^{n-1}$.

We now take a circular basis of such $V$, i.e. $\{e_1, \cdots, e_{2n+1}\} \subseteq V$ such that

$$\langle e_i, e_j \rangle = \begin{cases} 1, & i-j \equiv \pm 1 \pmod{2n+1} \\ 0, & \text{otherwise.} \end{cases}$$

Note that such subset satisfies $\sum_{i=1}^{2n+1} e_i = 0$

For any $i \in \{1, \cdots, 2n+1\}$, we set $V_i := \frac{\langle e_i \rangle^\perp}{\langle e_i \rangle}$, it is an $(2n-2)$-dimensional symplectic space with induced circular basis. (For example, $V_3$ has circular basis $e_1, e_2+e_3+e_4, e_5, \cdots, e_{2n+1}$).

We also construct a class $\mathcal{X}(V)$ of subspaces of $V$ by induction on $n$: A subspace $X \subseteq V$ is in $\mathcal{X}(V)$ if:

$$\begin{cases} \text{either } X = 0; \\ \text{or } X \text{ is the inverse image of some } X' \in \mathcal{X}(V_i) \\ \quad \text{under the projection } \langle e_i \rangle^\perp \twoheadrightarrow V_i. \end{cases}$$

One can check that $\mathcal{X}(V)$ consists of isotropic subspaces of $V$.

We now have the following theorem:

Theorem 1 1) The set $B := \{ b_X \mid X \in \mathcal{X}(V) \}$ is a basis of $\mathbb{C}^V$. In particular $|B| = |V|$.

2) There exists a bijection $V \to B$, $v \mapsto b^v$ such that $b^v(v) \neq 0$ for all $v \in V$.

3) Under suitable order, the matrix that converts $B$ to the basis $\{b_{\{x\}} \mid x \in V\}$ is upper triangular with diagonal entries 1.

4) For any $b_X \in B$, $\hat{b}_X$ has form

$$\hat{b}_X = \pm b_X + \sum_{\substack{X' \in \mathcal{X}(V) \\ \dim X' > \dim X}} c_{X,X'} b_{X'},$$

where $c_{X,X'}$ is a certain complex number.

In other words, under suitable order of $B$, the matrix of the map $f \mapsto \hat{f}$ under $B$ is upper triangular with diagonal entries $\pm 1$.

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We now give related results on non-abelian Fourier transforms.

Suppose that $\Gamma$ is a finite group. We define

$$M(\Gamma) := \left\{ (x,\sigma) \,\middle|\, \begin{array}{l} x \in \Gamma \\ \sigma \text{ is an irreducible representation of } Z(x) \end{array} \right\} \Big/ \sim ,$$

where $(x,\sigma) \sim (x',\sigma')$ if and only if there exists $g \in \Gamma$ such that $gxg^{-1} = x'$, $g\sigma g^{-1} = \sigma'$.

We set $\mathbb{C}[M(\Gamma)]$ to be the $\mathbb{C}$-vector space with basis $M(\Gamma)$.

We define a pairing $\{-,-\} : \mathbb{C}[M(\Gamma)] \times \mathbb{C}[M(\Gamma)] \to \mathbb{C}$ by

$$\{(x,\sigma),(x',\sigma')\} = \sum \underline{\mathrm{tr}(gxg^{-1},\sigma')\,\mathrm{tr}(g^{-1}x'g,\sigma)}$$

$$\sum_{\substack{g\in\Gamma \\ gxg^{-1}x'=x'gxg^{-1}}} \quad |Z(x)|\,|Z(x')|$$

We now define the non-abelian Fourier transform by

$$A: \mathbb{C}[M(\Gamma)] \to \mathbb{C}[M(\Gamma)],$$

$$(x,\sigma) \mapsto \sum_{x',\sigma'} \{(x,\sigma),(x',\sigma')\}(x',\sigma').$$

$A$ has the property that $A^2=1$.

We say that an element $b\in\mathbb{C}[M(\Gamma)]$ is positive if $b$ is a linear combination of $(x,\sigma)$'s with nonnegative coeffecients; $b$ is bipositive if both $b$ and $A(b)$ are positive.

We now have the following theorem:

**Theorem 2.** Suppose that $\Gamma$ has form $S_2\times S_2\times\cdots\times S_2$, $S_3$, $S_4$ or $S_5$. Then there exists a basis $B$ of $\mathbb{C}[M(\Gamma)]$ such that:

0) $B$ consists of bipositive elements;

1) The matrix that converts $B$ to the basis $M(\Gamma)$ is upper triangular with diagonal entries 1;

2) The matrix of $A: \mathbb{C}[M(\Gamma)]\to\mathbb{C}[M(\Gamma)]$ under the basis $B$ is upper triangular with diagonal entries $\pm 1$.

⌈Note that Theorem 2 generalizes Theorem 1, since in the case $\Gamma=\prod_{i=1}^{n} S_2$, the set $M(\Gamma)$ is canonically an $\mathbb{F}_2$-vector space, with a symplectic form $\langle -,- \rangle$ such that $(-1)^{\langle b_1, b_2 \rangle} = \{b_1, b_2\}$. thus the Fourier transform in Theorem 1 coincides with A. $\dashv$

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At the end of the note we point out that the basis B can be related to certain set of classes of intervals in $\{1, \cdots, 2n\}$, where here an interval of the form $[a, b]$ means $\{a, a+1, \cdots, b-1, b\}$. The correspondence in the case $n=2$ is as follows:

| class of intervals in $\{1,2,3,4\}$ | | isotropic subspace |
|---|---|---|
| $\emptyset$ | $\longmapsto$ | $(0)$ |
| $\{1\}$ | $\longmapsto$ | $(0, \boxed{1})$ |
| $\{2\}$ | $\longmapsto$ | $(0, \boxed{2})$ |
| $\{3\}$ | $\longmapsto$ | $(0, \boxed{3})$ |
| $\{4\}$ | $\longmapsto$ | $(0, \boxed{4})$ |
| $\{1, 3\}$ | $\longmapsto$ | $(0, 1, 3, \boxed{13})$ |
| $\{1, 4\}$ | $\longmapsto$ | $(0, 1, 4, \boxed{14})$ |
| $\{2, 4\}$ | $\longmapsto$ | $(0, 2, 4, \boxed{24})$ |
| $\{2, 123\}$ | $\longmapsto$ | $(0, 2, 13, \boxed{123})$ |
| $\{3, 234\}$ | $\longmapsto$ | $(0, 3, 24, \boxed{234})$ |
| $\{1234\}$ | $\longmapsto$ | $(0, \boxed{1234})$ |
| $\{3, 1234\}$ | $\longmapsto$ | $(0, 3, 1234, \boxed{124})$ |
| $\{2, 1234\}$ | $\longmapsto$ | $(0, 2, 1234, \boxed{134})$ |
| $\{4, 12\}$ | $\longmapsto$ | $(0, 4, 124, \boxed{12})$ |
| $\{1, 34\}$ | $\longmapsto$ | $(0, 1, 134, \boxed{34})$ |

$$\{1234,\ 12\} \quad \longmapsto \quad (0,\ 1234,\ 14,\ \boxed{23}).$$

Here the number sequences $(a, a+1, \cdots, b-1, b)$ on the left hand side are intervals in $\{1,2,3,4\}$; the number sequences $(i_1, \cdots, i_n)$ on the right hand side are elements $e_{i_1} + \cdots + e_{i_n} \in V$, where $\{e_1, \cdots, e_5\}$ is a circular basis. Boxed elements are elements that do not appear in previous isotropic subspaces. The correspondence is given by:

For any $\{I_1, \cdots, I_k\}$ on the left hand side, it corresponds to the subspace of $V$ spanned by $\sum_{i \in I_r} e_i$, for $r = 1, \cdots, k$.

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Related references:

- Lusztig, "The Grothendieck group of unipotent representations: A new basis";
- Lusztig, "Fourier transform as a triangular matrix";
- Lusztig, "Unipotent representations of a finite Chevalley group of type $E_8$".