Lecture No 8 April 1, 2022 (Fri)

② Upcrossing inequality

$X = (X_n)$: $(\mathcal{F}_n)$-adapted stochastic process. Take $a, b \in \mathbb{R}$ such that $a < b$ and define

$$
\begin{aligned}
\sigma_1 &= \min\{n \geq 1; X_n \leq a\} \\
\sigma_2 &= \min\{n > \sigma_1; X_n \geq b\} \\
&\cdots \\
&\cdots \\
\sigma_{2k+1} &= \min\{n > \sigma_{2k}; X_n \leq a\} \\
\sigma_{2k+2} &= \min\{n > \sigma_{2k+1}; X_n \geq b\}, \\
&\cdots \\
&\cdots
\end{aligned}
$$

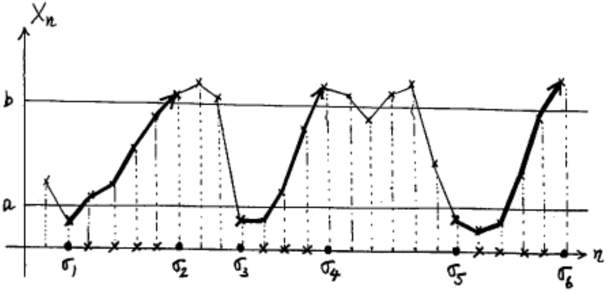
for $k = 1, 2, \ldots$, where $\min \emptyset =: \infty$. Then, $(\sigma_n)_{n=1,2,\ldots}$ is an increasing sequence of Markov times:

$$\sigma_1 < \sigma_2 < \sigma_3 < \cdots \qquad (\text{if } \sigma_n < \infty)$$

σ_1: The first time that $X \leq a$ happens.
σ_2: After σ_1, the first time that $X \geq b$ happens.
σ_3: After σ_2, the first time that $X \leq a$ happens again.
σ_4: After σ_3, the first time that $X \geq b$ happens again.
We continue to define these times repeatedly. Then, set

$$U_n := \max\{k; \sigma_{2k} \leq n\} \in \mathbb{Z}_+(\equiv \{0\} \cup \mathbb{N}),$$

where $\max \emptyset = 0$. U_n, called upcrossing number (for $a \nearrow b$), counts how many times X crosses the interval $[a, b]$ from below a to above b in time interval $[1, n]$.



We give an upper bound for the expectation of U_n to show that a submartingale X cannot fluctuate many times.

[Theorem 9.8] If (X_n) is a submartingale, we have

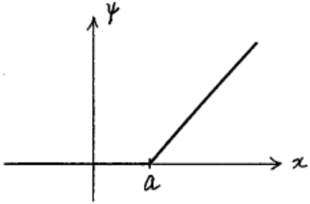
$$(b-a)E[U_n] \leq E[(X_n-a)^+] \qquad \square$$

[Proof] [Step 1] $(Y_n := (X_n-a)^+)_{n=1,2,\ldots}$ is a submartingale

$\because$ $\psi(x) = (x-a)^+, x \in \mathbb{R}$ is convex and increasing. Thus,

$$Y_n = \psi(X_n) \leq \psi\big(E[X_{n+1}|\mathcal{F}_n]\big) \leq E[\psi(X_{n+1})|\mathcal{F}_n] = E[Y_{n+1}|\mathcal{F}_n]$$

1st inequality follows from "X: submartingale and ψ: increasing", and 2nd follows from Jensen's inequality. $\square$



[Step 2] For $n = 2, 3, \ldots$, set

$$H_n = \begin{cases} 1, & \text{if } {}^{\exists}k \geq 1 \text{ s.t. } \sigma_{2k-1} < n \leq \sigma_{2k} \\ 0, & \text{if no such } k \text{ exists.} \end{cases}$$

i.e., We count 1 when X is in upcrossing state after it takes values $\leq a$. Then, H is predictable.

⛶ From $\{H_n = 1\} = \bigcup_{k=1}^{\infty}\{\sigma_{2k-1} < n \leq \sigma_{2k}\}$ and $\{\sigma_{2k-1} < n \leq \sigma_{2k}\} = \{\sigma_{2k-1} \leq n-1\} \cap \{\sigma_{2k} \leq n-1\}^c \in \mathcal{F}_{n-1}$, we see $\{H_n = 1\} \in \mathcal{F}_{n-1}$. □

[Step 3] Let us prove $(b-a)U_n \leq (H \cdot Y)_n$.

When $n=1$, since $U_1 = 0 = (H \cdot Y)_1$, this is obvious.
When $n \geq 2$, RHS is

$$(H \cdot Y)_n = \sum_{k=2}^{n} H_k (Y_k - Y_{k-1}).$$

Recalling that (H_k) takes value 1 only when (X_k) is in upcrossing state, we can further rewrite $(H \cdot Y)_n$ as

$$= \sum_{\substack{2 \leq k \leq n: \\ {}^{\exists} i \geq 1 \text{ s.t. } \sigma_{2i-1} < k \leq \sigma_{2i}}} (Y_k - Y_{k-1})$$

$$= \sum_{i=1}^{U_n} \left(Y_{\sigma_{2i}} - Y_{\sigma_{2i-1}} \right) + \left(Y_n - Y_{\sigma_{2U_n+1} \wedge n} \right).$$

Note that $\sigma_{2U_n+2} > n$. (For example from the above figure,
$n \in [\sigma_3, \sigma_4) \Rightarrow U_n = 1 \Rightarrow \sigma_{2U_n+2} = \sigma_4$, while
$n \in [\sigma_4, \sigma_5) \Rightarrow U_n = 2 \Rightarrow \sigma_{2U_n+2} = \sigma_6$, and
in both cases, $n < \sigma_{2U_n+2}$ holds.)

However, in the sum of 1st term,

$$Y_{\sigma_{2i}} \geq b - a, \quad Y_{\sigma_{2i-1}} = 0$$

On the other hand, for the 2nd term, set

$$Z_n = Y_n - Y_{\sigma_{2U_n+1} \wedge n}.$$

Then, if $\sigma_{2U_n+1} \geq n$,

$$Z_n = Y_n - Y_n = 0$$

while if $\sigma_{2U_n+1} < n$, $Y_{\sigma_{2U_n+1} \wedge n} = Y_{\sigma_{2U_n+1}} = 0$ so that

$$Z_n = Y_n \geq 0.$$

Thus, for both cases, we have $Z_n \geq 0$. Accordingly, we obtain

$$(H \cdot Y)_n \geq \sum_{i=1}^{U_n} (b - a) = (b - a) U_n.$$

Step 3 is concluded. □

[Step 4] $K_n := 1 - H_n$ is predictable and takes values 0 or 1. Therefore, by Theorem 9.7-(2), $((K \cdot Y)_n)$ is a submartingale. However, since $H_k + K_k = 1$, we have

$$\begin{aligned} Y_n - Y_1 &= \sum_{k=2}^{n} (H_k + K_k)(Y_k - Y_{k-1}) \\ &= (H \cdot Y)_n + (K \cdot Y)_n. \end{aligned}$$

Thus, by Step 3, we have

$$E[(b-a)U_n] \leq E[(H \cdot Y)_n] = E[Y_n - Y_1 - (K \cdot Y)_n].$$

In this estimate, since $((K \cdot Y)_n)$ is a submartingale and $(K \cdot Y)_1 = 0$, we have

$$E[(K \cdot Y)_n] \geq E[(K \cdot Y)_1] = 0.$$

Moreover, noting $Y_1 = (X_1 - a)^+ \geq 0$, we finally obtain

$$E[(b-a)U_n] \leq E[Y_n]$$

and this shows the conclusion of Theorem 9.8. □

③ Submartingale convergence theorem

If an increasing real sequence (a_n) is bounded above, the limit $\lim_{n\to\infty} a_n$ exists. The next theorem is, in a sense, a probabilistic version of this statement. For the proof, we use the upcrossing inequality.

[Theorem 9.9] If a submartingale (X_n) satisfies

$$\sup_n E[X_n^+] < \infty,$$

then X_n converges in a.s.-sense to a certain r.v. X (i.e., $X = \lim_{n\to\infty} X_n$ a.s.) and the limit X is integrable (i.e., $E[|X|] < \infty$). □

[Proof] [Step1] Fix $^\forall a < b$ and denote by $U_n^{a,b}$ the upcrossing number of X for $a \nearrow b$ in time interval $[1, n]$. Since it is increasing in n, the limit $U_{a,b} := \lim_{n\to\infty} U_n^{a,b} \in \mathbb{Z}_+ \cup \{\infty\}$ exists. However, by Theorem 9.8, one can estimate as

$$E[U_n^{a,b}] \leq \frac{1}{b-a} E[(X_n - a)^+] \\ \leq \frac{1}{b-a} \left(E[X_n^+] + |a| \right),$$

which is bounded in n by our assumption. Therefore, by monotone convergence theorem, we have

$$E[U^{a,b}] = \lim_{n\to\infty} E[U_n^{a,b}] < \infty$$

Thus, we have shown $U^{a,b} < \infty$ a.s. for $^\forall a < b$.

[Step 2] Next,

$$P\left\{\liminf_{n\to\infty} X_n \neq \limsup_{n\to\infty} X_n\right\}$$

$$= P\left\{\bigcup_{\substack{a,b\in\mathbb{Q}\\ a<b}} \left(\liminf_{n\to\infty} X_n < a < b < \limsup_{n\to\infty} X_n\right)\right\}$$

$$= P\left\{\bigcup_{\substack{a,b\in\mathbb{Q}\\ a<b}} \left(U^{a,b} = \infty\right)\right\} \leq \sum_{\substack{a,b\in\mathbb{Q}\\ a<b}} P\left(U^{a,b} = \infty\right) \underset{\text{Step 1}}{=} 0$$

$\therefore$ $P\left(\liminf_{n\to\infty} X_n = \limsup_{n\to\infty} X_n\right) = 1$

$\therefore$ We see ${}^{\exists}X = \lim_{n\to\infty} X_n \in \mathbb{R} \cup \{\pm\infty\}$ a.s.

[Step 3] Finally, we show that the limit X satisfies $X \in \mathbb{R}$ a.s. First, by Fatou's lemma and the assumption of the theorem, we have

$$E[X^+] \le \underline{\lim}_{n\to\infty} E[X_n^+] < \infty.$$

On the other hand, again by Fatou's lemma,

$$E[X^-] \le \underline{\lim}_{n\to\infty} E[X_n^-].$$

However, since X is a submartingale, we have $E[X_n] \ge E[X_1]$ and therefore

$$E[X_n^-] = E[X_n^+] - E[X_n] \le E[X_n^+] - E[X_1].$$

$\therefore$ We obtain

$$E[X^-] \le \underline{\lim}_{n\to\infty} E[X_n^+] - E[X_1] < \infty$$

We have shown that both $E[X^+], E[X^-]$ are finite so that

$$E[|X|] < \infty.$$

This implies $|X| < \infty$ *a.s.*. □

Summary of discussions for martingales by now:

Setting:

$(\Omega, \mathcal{F}, P)$: Probability space
$(\mathcal{F}_n)_{n=1,2,\dots}$: filtration (or reference family)
$X = (X_n)_{n=1,2,\dots}$: $(\mathcal{F}_n)$-adapted stochastic process

Discussed and showed:

9.1 Definition of discrete time martingales and submartingales
9.2 Doob decomposition
9.3 Markov time
9.4 Doob's optional sampling theorem
9.5 Doob's inequality
9.6 Submartingale convergence theorem

9.7 Moment inequality

In §9.5, we showed the following Doob's inequality:

[Theorem 9.6] (X_n): submartingale,

$$(1) \quad aP\left(\max_{1\leq k\leq n} X_k \geq a\right) \leq E\left[X_n, \max_{1\leq k\leq n} X_k \geq a\right] \leq E[X_n^+]$$

for $^\forall a > 0$. □

[Corollary] (M_n): pth power integrable $(p \geq 1)$ martingale

$$P\left(\max_{1\leq k\leq n} |M_k| \geq a\right) \leq \frac{1}{a^p} E\left[|M_n|^p\right], \quad {}^\forall a > 0.$$ □

Here, we rewrite these inequalities in more convenient pth moment estimates.

(1) Variant of Doob's inequality

[Theorem 9.10] (Doob) Let $p > 1$ and let (M_n) be a pth power integrable martingale. Then, we have

$$E\left[\max_{1\leq k\leq n} |M_k|^p\right] \leq \left(\frac{p}{p-1}\right)^p E\left[|M_n|^p\right]$$

□

[Proof] Set $Y = \max_{1 \le k \le n} |M_k|$ (note $E[Y^p] < \infty$), then

$$
\begin{aligned}
E[Y^p] &= \int_\Omega Y^p dP = \int_\Omega dP \int_0^Y p a^{p-1}\, da \\
&= \int_\Omega dP \int_0^\infty p a^{p-1} 1_{\{Y \ge a\}}\, da \\
&\underset{\text{Fubini}}{=} \int_0^\infty p a^{p-1} da \int_\Omega 1_{\{Y \ge a\}} dP
\end{aligned}
$$

Since $(|M_k|)$ is a submartingale, by Doob's inequality (1st inequality in Theorem 9.6 (1)), we have

$$
\int_\Omega 1_{\{Y \ge a\}} dP = P(Y \ge a) \le \tfrac{1}{a} E\left[|M_n|, Y \ge a\right].
$$

Thus,

$$
\begin{aligned}
E[Y^p] &\le p \int_0^\infty a^{p-2}\, da \int_\Omega |M_n| 1_{\{Y \ge a\}}\, dP \\
&\underset{\text{Fubini}}{=} p \int_\Omega |M_n|\, dP \int_0^\infty a^{p-2} 1_{\{Y \ge a\}}\, da
\end{aligned}
$$

However,

$$\int_0^\infty a^{p-2} 1_{\{Y \geq a\}}\, da = \int_0^Y a^{p-2}\, da = \frac{Y^{p-1}}{p-1}.$$

Therefore,

$$\begin{aligned} E[Y^p] &\leq \frac{p}{p-1} E\left[|M_n| Y^{p-1}\right] \\ &\underset{\text{Hölder}}{\leq} \frac{p}{p-1} E\left[|M_n|^p\right]^{\frac{1}{p}} \cdot E\left[Y^p\right]^{\frac{p-1}{p}} \end{aligned}$$

Divide both sides by $E\left[Y^p\right]^{\frac{p-1}{p}}$, and take pth power of them. Then we obtain the conclusion. □

(2) Burkholder's inequality

$(M_n)_{n=0,1,2,\dots}$: square-integrable martingale s.t. $M_0 = 0$.
In particular, $E[M_n] = 0$ for $^\forall n$.

[Definition] We call

$$[M]_n = \sum_{k=1}^{n} (M_k - M_{k-1})^2, \quad n = 1, 2, \dots$$

a quadratic variation of (M_n). □

- Quadratic variation $[M]_n$ has the following 2 properties.

(1) $(M_n^2 - [M]_n)_{n=1,2,\dots}$ is a martingale.

(2) $([M]_n)$ is increasing (precisely, non-decreasing) process.

⚆ (2) is clear from Definition. To prove (1), we rewrite

$$M_n^2 - [M]_n = \sum_{k=1}^{n}(M_k^2 - M_{k-1}^2) - \sum_{k=1}^{n}(M_k - M_{k-1})^2$$
$$= 2\sum_{k=1}^{n} M_{k-1}(M_k - M_{k-1}).$$

Here, since M_{k-1} is predictable (i.e. $\mathcal{F}_{k-1}$–measurable), RHS is the form of martingale transform and we obtain (1). □

Since $(M_n^2)_{n=1,2,\ldots}$ is a submartingale, it has Doob decomposition:

1) $M_n^2 = N_n + A_n, n = 1, 2, \ldots$
2) $(N_n)_{n=1,2,\ldots}$ is a martingale
3) $(A_n)_{n=1,2,\ldots}$ is a predictable increasing process s.t. $A_1 = 0$

From 1), 2), we see

1)′ $(M_n^2 - A_n)$ is a martingale

In particular, (A_n) has the same properties (1), (2) stated above for $[M]_n$.

[Definition] We denote (A_n) by $(\langle M \rangle_n)$ and call it again the quadratic variation of (M_n). □

[Remark] $\langle M \rangle_n$ is predictable, but $[M]_n$ is not. We must distinguish these two definitions, though they are called by the same name. In fact, for continuous time continuous martingales $(M_t)_{t \geq 0}$ (second "continuous" means M_t is continuous in t), these two processes coincide.

[Theorem 9.11] **(Burkholder–Davis–Gundy's inequality)**
For $^{\forall}p \geq 1$, $^{\exists}c_p, C_p > 0$ s.t. the following bound holds for $(M_n)_{n=0,1,2,\ldots}$: pth power integrable martingale s.t. $M_0 = 0$,

$$c_p E\left[[M]_n^{\frac{p}{2}}\right] \leq E\left[\max_{1\leq k\leq n}|M_k|^p\right] \leq C_p E\left[[M]_n^{\frac{p}{2}}\right]. \qquad \square$$

[Remark] The upper bound is useful for application.

• The upper bound for $p = 2$ follows from Theorem 9.10, a variant of Doob's inequality.

∵

$$E\left[\max_{1\leq k\leq n}|M_k|^2\right] \underset{\text{Doob}}{\leq} 4E[M_n^2] = 4E\left[[M]_n\right] \qquad \square$$

[Remark] For continuous time martingales, a simple proof based on Itô's formula is known. See [Karatzas-Shreve] or Ikeda-Watanabe "SDEs and Diffusion Processes".

• Here, as a guide for $p > 2, p \in \mathbb{Z}$, we only give a simple proof of the upper bound for $p = 4$. Rewrite

$$
\begin{aligned}
M_n^4 \underset{M_0=0}{=} & \sum_{k=1}^{n} (M_k^4 - M_{k-1}^4) \\
&= \sum_{k=1}^{n} (M_k^3 + M_k^2 M_{k-1} + M_k M_{k-1}^2 + M_{k-1}^3)(M_k - M_{k-1}) \\
&= \sum_{k=1}^{n} (M_k^2 + 2M_k M_{k-1} + 3M_{k-1}^2)(M_k - M_{k-1})^2 \\
&\quad + 4 \sum_{k=1}^{n} M_{k-1}^3 (M_k - M_{k-1}).
\end{aligned}
$$

The 2nd term has mean 0, since it is a martingale transform. (Integrability is fine, though we assumed the boundedness of H_k in Theorem 9.7.) For the 1st term, we estimate $2M_k M_{k-1} \le M_k^2 + M_{k-1}^2$ to obtain

$$E[M_n^4] \le E\left[6 \max_{1\le k\le n} M_k^2 \cdot \sum_{k=1}^{n}(M_k - M_{k-1})^2\right]$$

$$\underset{\text{Schwarz}}{\le} 6E\left[\max_{1\le k\le n} M_k^4\right]^{\frac{1}{2}} E[[M]_n^2]^{\frac{1}{2}}$$

Here, we have used $\sum_{k=1}^{n}(M_k - M_{k-1})^2 = [M]_n$. However, by a variant of Doob's inequality (Theorem 9.10),

$$E\left[\max_{1\le k\le n} M_k^4\right] \le (\tfrac{4}{3})^4 E[M_n^4].$$

Use this for the LHS of the above estimate and divide the both sides by $E\left[\max_{1\le k\le n} M_k^4\right]^{\frac{1}{2}}$. Then, we obtain the desired upper bound for $p = 4$. □