2020-11-24 Kähler geometry (Taxing Lin)

A variational approach to the Yau-Tian-Donaldson conj.

— Berman-Boucksom-Jonsson

arXiv: 1509.04561. v3

We focus on automorphism gp is discrete because coercivity of the Ding functional forces the automorphism gp to be discrete. (c.f. Thm 5.4. [BBEGZ]. 16
Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties.)

$$\begin{array}{ccc}
\text{existence of KE} & & \text{uniformly } K\text{-stable} \\
\text{Tian } \Big\Updownarrow \text{ Darvas-Rubinstein.} & & \text{MMP } \Big\Updownarrow \text{ v1. of BBJ} \quad (\text{i.e. } M^{NA} \text{ is coercive}) \\
\text{coercivity of Ding functional} & \Longleftrightarrow & \text{uniformly Ding-stable} \\
& & (\text{i.e. } D^{NA} \text{ is coercive})
\end{array}$$

§1. finite energy potential and Psh geodesic.

§2. twisted KE current and coercivity.

§3. valuations and stability.

§4. Psh rays and Lelong number

§5. Ding stability and twisted KE.

§6. Non-Archimedean potentials of finite energy and maximal geodesic rays.

I. Preliminaries (pluripotential)

·1. finite energy potential. (X, ω_0) cpt. Kähler mfd

$PSH := PSH(X, \omega_0) =$ the space of ω_0-psh function

$u \in PSH$ i.e. u. u.s.c. and $\omega_0 + dd^c u \geq 0$ in the sense of current.

with natural weak topology, coincides with the L^1-top.

By Hartogs lemma, the functional $u \longmapsto \sup_X u$ is cont.

and the space

$$PSH_{\sup} := \{ u \in PSH \mid \sup_X u = 0 \}$$

is cpt.

By regularization thm [Błocki–Kołodziej. 07],

every $u \in PSH$, $\exists u_k \in \mathcal{H} := \{ u \in C^\infty(X) \mid \omega_u := \omega_0 + dd^c u > 0 \}$

s.t. $u_k \searrow u$.

- Monge-Ampère energy

$$E: \mathcal{H} \longrightarrow \mathbb{R}$$
$$u \longmapsto E(u) = \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X u\, \omega_u^j \wedge \omega_0^{n-j}$$

$$DE_u(\delta u) = V^{-1} \int_X \delta u\, \omega_u^n \qquad u \in \mathcal{H},\ \delta v \in C^\infty(M)$$

$$V = \int_X \omega_0^n$$

<u>Def</u>: $MA(u) := V^{-1} \omega_u^n$, $u \in \mathcal{H}$, it's a probability measure.

$\forall u, v \in \mathcal{H}$, $E(u) - E(v) = \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X (u-v)\, \omega_u^j \wedge \omega_v^{n-j}$

- $E(u+c) = E(u) + c$, $c \in \mathbb{R}$
- $u \leq v \Rightarrow E(u) \leq E(v)$, with equality iff $u = v$

E admits a unique extension as a monotone, u.s.c. functional,

$$E: PSH \longrightarrow \mathbb{R}$$
$$u \longmapsto E(u) := \inf\{E(v);\ v \geq u,\ v \in \mathcal{H}\}$$

<u>finite energy potential</u>:

$$\mathcal{E}^1 = \mathcal{E}^1(X, \omega_0) := \{u \in PSH,\ E(u) > -\infty\}$$

$$\Sigma'_{\sup} := \Sigma' \cap PSH_{\sup} = \{ u \in \Sigma',\ \sup_X u = 0 \}$$

- Def: strong topology on Σ'. the coarsest refinement of the weak topology in which $E : \Sigma' \to \mathbb{R}$ become cont.

i.e. $\varphi_j \xrightarrow[\text{strong}]{} \varphi$ in Σ' iff. $\varphi_j \to \varphi$ & $E(\varphi_j) \to E(\varphi)$.

By [BBEGZ, 16]. $u_0, \cdots, u_n \in \Sigma' \longmapsto \int_X u_0 \omega_{u_1} \wedge \cdots \wedge \omega_{u_n}$ are well-defined, and cont. w.r.t. $(u_0, \cdots, u_n)$ in the strong topology.

I.2 Psh Path

Consider. $\mathbb{D}_I := \{ \tau \in \mathbb{C}^* \mid -\log|\tau| \in I \}$ $I \subseteq \mathbb{R}$ an interval

identify map $U : I \to PSH$ with S^1-invariant function on $X \times D_I$, defined by $U_t(x) =: U(x.\tau)$ $t = -\log|\tau|$

Def: A psh path is a map $U : I \to PSH$ (I. open). s.t. the corresponding function on $X \times \mathbb{D}_I$ is $p_1^* \omega$-Psh, where $p_1 : X \times \mathbb{C} \to X$.

$\Rightarrow t \mapsto U_t(x)$ is convex on I for each fixed $x \in X$, and hence admits an limit in $[-\infty, +\infty]$ as t tends to ∂I

Prop 1.5. For any psh path $U: I \to PSH$, then $\mathrm{Im}(U) \cap \mathcal{E}^1 = \phi$ or $\mathrm{Im}(U) \subseteq \mathcal{E}^1$, In the latter case, $U: I \to \mathcal{E}^1$ is cont. (in the strong topology) and $t \mapsto E(U_t)$ is convex

I.3. Psh geodesics

Psh Path $W: (0,1) \to PSH$ is dominated by ω-psh $u_0, u_1 \in PSH$ if

$$\lim_{t \to 0} W_t \leq u_0, \quad \lim_{t \to 1} W_t \leq u_1$$

If such W exists, a simple envelope argument show that there exists a largest one. $U: (0,1) \to PSH$, called Psh geodesic joining u_0 to u_1

$$(*) \qquad U = \sup_{W \in S} W \qquad S := \{\text{Psh Path } W \text{ s.t. } \lim_{t \to 0,1} W_t \leq u_{0,1}\}$$

Lemma. when $u_0, u_1 \in PSH \cap L^\infty$, then such U given by $(*)$ is the unique bounded $p_1^* \omega_0$-psh solution of the equation

$$\begin{cases} (p_1^*\omega_0 + dd^c U)^{n+1} = 0 \\ \omega_0 + dd^c U|_t > 0, \quad U \text{ is } S^1\text{-inv.} \\ \lim_{t\to 0} U_t = u_0 \text{ and } \lim_{t\to 1} U_t = u_1 \end{cases}$$

Rmk: • when $u \in C^\infty(X \times D_{[0,1]})$, this is a observation of Semmes [92] and Donaldson [99].

U_t is the geodesic in the sense of Riem. geometry w.r.t. Riem. structure of Mabuchi and Donaldson of $\mathcal{H}$.

This is why we call the U_t is psh geodesic

• the supremum of a family of $p_1^*\omega_0$-psh functions may not be $p_1^*\omega_0$ psh. but such U is also $p_1^*\omega_0$-psh

Indeed, by convexity in the t, $\forall W \in S$, satisfies

$W_t (= W(\cdot, t)) \le (1-t)u_0 + tu_1$

taking supremum on L.S.H. $\Rightarrow U_t \le (1-t)u_0 + tu_1$

By taking the u.s.c. regularization of the above ineq.

$$\Rightarrow U_t^* \le (1-t)u_0 + tu_1$$

so $\Rightarrow U^* \in S$, hence $U^* \le U$ by def. of U

$u \le u^*$ $\Longrightarrow$ $u = u^* \in PSH(p_1^*\omega_0, X \times \mathbb{D}_I)$.

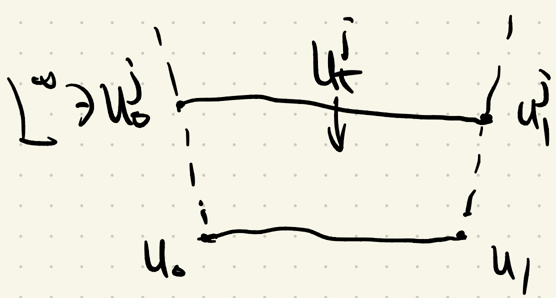
- when $u_0, u_1 \in \mathcal{H}$, By Chen's result [00], refined by Błocki, the psh geodesic joining them as a $C^{1,1}$ function on $X \times \mathbb{D}$. ($u \in L^1_{loc}$)

when $u_0, u_1 \in \mathcal{E}^1$. By Darvas's work, the psh geodesic joining them exists and yields a const. speed geodesic in the Darvas metric

Thm 1.7 For any $u_0, u_1 \in \mathcal{E}^1$, the psh geodesic joining them exists and defines a continuous $u : [0,1] \to \mathcal{E}^1$ with $E(u_t)$ affine on $[0,1]$.

Conversely, any cont. path $\tilde{u} : [0,1] \to \mathcal{E}^1$ joining u_0 and u_1 with $E(\tilde{u}_t)$ affine and $\tilde{u}$ psh on $(0,1)$ satisfies

$$\tilde{u} = u.$$



Cor 1.8. For $u: I \to \mathcal{E}^1$ (I not nec. open or bdd). TFAE.

(i). the restriction of u to each cpt interval $[a,b] \subset I$ coincide (up to affine reparametrization) with the psh geodesic joining u_a to u_b

(ii). u is strongly cont. on I, psh on the $\mathring{I}$, and $E(u_t)$ is affine on I.

Def 1.9. $u: I \to \mathcal{E}^1$ satisfying the equivalent conditions of Cor 1.8 is called psh geodesic in $\mathcal{E}^1$.

A psh geodesic ray is a psh geodesic $u: \mathbb{R}_{>0} \to \mathcal{E}^1$

I.4. Darvas's d_1-distance.

The strong topology is defined by the metric

$$d(u,v) = \|u-v\|_{L^1(\omega_0)} + |E(u) - E(v)| \quad \forall u,v \in \mathcal{E}^1$$

By Darvas's work, he construct a natural L^1-Finsler metric d_1 on $\mathcal{H}$, s.t. $\mathcal{E}^1 = \overline{\mathcal{H}}^{d_1}$

$$d_1(u,v) = \inf \left\{ \int_0^1 \|\dot{u}_t\|_{L^1(MA(u_t))} dt \;\middle|\; (u_t)_{t\in[0,1]} \text{ is} \right.$$

sm. path joining u to v $\}$

If $u_0, u_1 \in \mathcal{E}^1$. $\exists$ decreasing seq. $u_0^{(k)}, u_1^{(k)} \in \mathcal{H}$ s.t. $u_0^{(k)} \searrow u_0$ and $u_1^{(k)} \searrow u_1$.

$$d_1(u_0, u_1) = \lim_{k\to\infty} d_1(u_0^{(k)}, u_1^{(k)})$$

If $u, v \in \mathcal{E}^1$ and $u \geq v$, $d_1(u,v) = E(u) - E(v)$.

In particular, $d_1(u, 0) = E(u)$ when $u \in \mathcal{E}^1$ and $\sup_X u = 0$

- Any psh geodesic $U : I \to \mathcal{E}^1$ in the sense of above Def. is a const. speed geodesic for d_1. i.e. $\exists C \geq 0$ s.t. $d_1(U_t, U_s) = c|t-s|$, $\forall t, s \in I$.

II. twisted KE current and coercivity.

As well known, smooth positive volume form μ on X $\xleftrightarrow{1:1}$ hermitian metric h on K_X^{-1} ($\underset{\nmid}{\overset{}{}} \Lambda^n \Omega^{1,0}$)

$$\mu = e^{-2f} i^{n^2} \Omega \wedge \bar{\Omega}$$

with $f = \log|\Omega|_h$, for local hol. volume form Ω.

- Ricci curvature of μ: $Ric(\mu) = dd^c f = dd^c \log|\Omega|_h$

$= \mathrm{Ric}(h)$

μ has <u>well-defined Ricci curvature</u> if $\mathrm{Ric}(\mu) = dd^c f$

in the sense of current.

Given a fixed closed (1,1)-current $\underline{\theta}$,

$$\mathrm{Ric}_\theta(\mu) := \mathrm{Ric}(\mu) - \theta \in c_1(X) - [\theta] =: c_1(X, \theta)$$

<u>Def 2.1</u>. θ-twisted KE current: $u \in \mathcal{E}^1$. s.t.

- ω_u^n has well-defined Ricci curvature.
- $\mathrm{Ric}_\theta(\omega_u) = \lambda\omega_u$. $\lambda \in \mathbb{R}$.

  ("$\mathrm{Ric}_\theta(\omega_u^n)$").

$\theta_0 \in [\theta]$ sm. form. $\psi \in L^1_{loc}$. s.t. $\theta = \theta_0 + dd^c\psi$ and

$\mathrm{Ric}(\omega_0) - \theta_0 = \lambda\omega_0 + dd^c \rho_0$, $\rho_0 \in C^\infty(X)$.

$$\mathrm{Ric}_\theta(\omega_u) = \lambda\omega_u \iff MA(u) = e^{2(\rho_0 - \lambda u - \psi + c)}\omega_0^n.$$

<u>Def 2.3</u>. A closed (1,1)-current θ is klt if θ is quasi-positive

i.e. $\theta = \theta_0 + dd^c\psi$. θ_0 sm. ψ: quasi-psh, and has

trivial multiplier ideal sheaf. i.e. $e^{-2\psi} \in L^1$

Guan-Zhou: $e^{-2\psi} \in L^p$. for some $p > 1$

Rem 2.4. θ quasi-positive current, $c_1(X, \theta) = \lambda [\omega_0]$, $\omega \in [\omega_0]$ twisted KE current. then

(i) if $\lambda \geq 0$, then θ is klt

(ii) if θ is klt. then ω has cont. potential, and is further a smooth Kähler form on any open set on which θ is smooth. (Thm 1.1. of BBEGZ.)