2021-10-13 Kähler geometry

Kähler case:

$$\frac{\omega^m}{m!} = \det(g_{i\bar{j}})\, i^m dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^m \wedge d\bar{z}^m$$

$$= dV_g$$

But we usually omit $m!$

and consider ω^m as the volume form.

$$\Delta_d = d^*d + dd^* \quad \text{on Riemannian}$$

$$\Delta_{\bar\partial} = \bar\partial^*\bar\partial + \bar\partial\bar\partial^* \quad \text{on Kähler}$$

$$\shortparallel$$

$$\Delta_\partial = \partial^*\partial + \partial\partial^*$$

Also $\Delta_d = \Delta_{\bar\partial} + \Delta_\partial = 2\Delta_{\bar\partial}$

$$\Delta_{\bar\partial} f = g^{i\bar j} \nabla_i \nabla_{\bar j} f = g^{i\bar j} \frac{\partial^2 f}{\partial z^i \partial \bar z^j}$$

$$= \frac{1}{2} \Delta_d f$$

More convenient to write $\Delta = \Delta_{\bar\partial}$

$$\int_M u \cdot \nabla_i X^i \omega^m = -\int \nabla_i u \cdot X^i \omega^m$$

$$= -\int X(u)\ \omega^m.$$

$\partial\bar{\partial} u = 0 \quad \Longrightarrow \quad \Delta u = 0. \Longrightarrow u = \text{const}$

$$0 = \int_M u \cdot \Delta u = -\int \nabla_i u \cdot \nabla^i u \ \omega^n$$

$$= -\int_M |\bar{\partial} u|^2 \omega^n$$

$\bar{\partial} u = 0 \qquad u = \text{const}.$

[Any harmonic function on a compact Riem mfd without boundary is constant]

Prescribing curvatures:

Typically there are two types of set ups.

1. Let M be a manifold (often compact).

Given a smooth function $F \in C^\infty(M)$, can we find a Riemannian metric g such that

Gaussian curvature is equal to F when $\dim M = 2$.

Scalar curvature is equal to F when $\dim M \geq 3$. ?

Particularly interesting is when F is constant.

2. Let (M, g) be a (compact) Riemannian manifold.

Given a smooth function $F \in C^\infty$ can we find a conformal metric $e^\sigma g$ such that

Gaussian curvature equal to F when $\dim M = 2$,

Scalar curvature equal to F when $\dim M \geq 3$.

In $\dim M = 2$ case, there is a natural restriction on F coming from Gauss-Bonnet theorem.

For example $F > 0$ is impossible if genus $(M) \geq 2$.

since $\chi(M) = \frac{1}{2\pi} \int F \, dv = 2 - 2g \leq 0$

There are many results. But the most famous one is:

Theorem (Yamabe problem (Schoen, Aubin))

Let (M, g_0) be a compact Riemannian manifold of dim $M \geq 3$. Then there is a conformal metric $e^{\sigma} g_0$ such that the scalar curvature of $e^{\sigma} g_0$ is a constant s.

Putting $e^{\sigma} = u^{\frac{4}{n-2}}$, $\gamma = \frac{4(n-1)}{n-2}$, $\alpha = \frac{n+2}{n-2}$,

the solution u must satisfy

$$-\gamma \Delta u + S_0 u = s u^{\alpha},$$

where s_0 is the scalar curvature of g_0.

There are topological obstruction to $s > 0$. (Hichin).

Most well-known unsolved question is

Nirenberg's problem

Let (S^2, g_0) be the unit sphere. Describe the set

$$G = \left\{ K \in C^{\infty}(M) \,\middle|\, \begin{array}{l} \exists\, e^{-\sigma} g_0 \text{ s.t. Gaussian curvature} \\ \text{of } e^{-\sigma} g_0 \text{ is } K \end{array} \right\}$$

The equation to be solved is

$$1 + \Delta \sigma = K e^{-\sigma}$$

Kazdan-Warner If $K \in G$ then

$$\int_{S^2} (\operatorname{grad} f) K \, e^{-\sigma} dV_g = 0$$

for f satisfying $\Delta f + 2f = 0$.

Going back to Monge-Ampere equation

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{-\varphi + F}$$

When $m=1$

$$1 + \Delta\varphi = e^F e^{-\varphi} = K e^{-\varphi} \qquad (K = e^F)$$

and has the same type as Nirenberg's problem.

Hinted by Kazdan-Warner's integrability condition I found an obstruction to the existence of KE metric on Fano manifolds.

Let M be a Fano manifold, $\omega \in c_1(M)$ a Kähler form. $\operatorname{Ric}(\omega) - \omega = \frac{i}{2\pi} \partial\bar\partial F$ for $\exists F \in C^\infty(M)$.
$\mathfrak{h}(M)$ the Lie algebra of all holomorphic vector field on M. $\operatorname{Fut}: \mathfrak{h}(M) \to \mathbb{C}$ defined by

$$\operatorname{Fut}(X) = \int_M XF \, \omega^m$$

is indep of $\omega \in c_1(M)$. If $\exists$ KE metric then $\operatorname{Fut} = 0$.