Principal Stratification

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Outline

- Potential outcomes and ACE
- Post-treatment variable
- Principal stratification
  - noncompliance
  - truncation by death
  - surrogate evaluation
- Identification and statistical inference
  - binary instrumental variable model
  - partial identification
  - principal ignorability
  - auxiliary independence
- Recent applications of principal stratification

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binary instrumental variable model
partial identification
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Recent applications of principal stratification

Potential outcome and average causal effect

- Observed data: treatment Z_i, outcome Y_i
- Potential outcomes: $Y_i(1)$ and $Y_i(0)$
- Observed outcome: $Y_i(Z_i) \rightsquigarrow$ only one potential outcome is observed for each unit

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- Individual causal effect: $Y_i(1) - Y_i(0) \rightsquigarrow$ difficult to estimate
- Average causal effect (ACE): $\mathbb{E}\{Y_i(1) - Y_i(0)\}$

Identification and inference

- Two methodological issues of causal inference:
    1. Identification: what can we learn if we have infinite amount of data? $\rightsquigarrow$ study design uncertainty
    2. Inference: what can we learn about identifiable quantities from a finite sample? $\rightsquigarrow$ statistical uncertainty

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- In order to achieve identification, assumptions are unavoidable, but we need to figure out what assumptions are plausible in practice $\rightsquigarrow$ design trumps analysis

Identification for ACE

- Randomized experiment: $Z_i \perp\!\!\!\perp Y_i(z)$

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- Unconfounded observational study: $Z_i \perp\!\!\!\perp Y_i(z) \mid \mathbf{X}_i$

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&= \mathbb{E}\left\{\frac{Z_i Y_i}{\mathbb{P}(Z_i = 1 \mid \mathbf{X}_i)}\right\} - \mathbb{E}\left\{\frac{(1 - Z_i) Y_i}{1 - \mathbb{P}(Z_i = 1 \mid \mathbf{X}_i)}\right\} \quad \text{(IPW)} \\
&= \mathbb{E}\left[\frac{Z_i\{Y_i - \mu_1(X_i)\}}{\mathbb{P}(Z_i = 1 \mid \mathbf{X}_i)}\right\} - \mathbb{E}\left\{\frac{(1 - Z_i)\{Y_i - \mu_0(X_i)\}}{1 - \mathbb{P}(Z_i = 1 \mid \mathbf{X}_i)}\right] \\
&\quad + \mathbb{E}\{\mu_1(X_i) - \mu_0(X_i)\} \quad \text{(doubly robust)}
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- Latent confounding: instrumental variable, DID, synthetic control, proximal inference...

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- Adjusting for the post-treatment variable is necessary
  - post-treatment variable encodes characteristics of the unit as well as of the treatment
  - ACE is not well defined, or cannot answer the question of interest

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  - criterion for a good surrogate: $Z \perp\!\!\!\perp Y \mid S$?

Issue of adjusting for observed post-treatment variable

- Potential values: $Y_i(z)$ and $S_i(z)$
- Observed variables: $S_i = S_i(Z_i)$ and $Y_i = Y_i(Z_i)$
- Randomized experiments: $\{Y_i(1), Y_i(0), S_i(1), S_i(0)\} \perp\!\!\!\perp Z_i$

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- A causal effect is defined to be the comparison between potential outcomes for the same units
  - Comparison of $\{Y_i(1), i \in \text{set}_1\}$ and $\{Y_i(0), i \in \text{set}_0\}$
  - Two sets should be identical: $\text{set}_1 = \text{set}_0$
  - ACE: $\mathbb{E}(Y_i \mid Z_i = 1) - \mathbb{E}(Y_i \mid Z_i = 0) = \mathbb{E}\{Y_i(1)\} - \mathbb{E}\{Y_i(0)\}$

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- Adjusting for the observed post-treatment variable does NOT yield valid causal quantities

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  - Larger $Y_i(z)$ for individuals with $S_i(0) = 1$
- Analysis using the survived individuals

$$
\begin{aligned}
& \mathbb{E}(Y_i \mid Z_i = 1, S_i = 1) - \mathbb{E}(Y_i \mid Z_i = 0, S_i = 1) \\
= \; & \mathbb{E}\{Y_i(1) \mid S_i(1) = 1\} - \mathbb{E}\{Y_i(0) \mid S_i(0) = 1\} \\
= \; & \mathbb{E}\{Y_i(0) \mid S_i(1) = 1\} - \mathbb{E}\{Y_i(0) \mid S_i(0) = 1\} \\
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- Hazard ratio has a built-in selection bias

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    - surrogate evaluation
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    - partial identification
    - principal ignorability
    - auxiliary independence
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Principal stratification

- Principal stratification: stratification based on the joint potential values $\{S_i(1), S_i(0)\}$, denoted by U_i
  - U_i is an unobserved variable
  - U_i is unaffected by the treatment, similar to a covariate
  - binary $S_i \rightsquigarrow$ four valued U_i
  - principal effect: comparison of $\{Y_i(1) : U_i = u\}$ and $\{Y_i(0) : U_i = u\}$

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- Any principal effect is a causal effect
  - principal causal effect: $\text{ACE}_u = \mathbb{E}\{Y_i(1) - Y_i(0) \mid U_i = u\}$
  - scientific meanings differ in different applications

$$\text{ACE} = \sum_u \mathbb{E}\{Y_i(1) - Y_i(0) \mid U_i = u\}\mathbb{P}(U_i = u)$$

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- Treatment effect: effect of the actually received treatment

Noncompliance

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- Four principal strata represent compliance behavior:
  - compliers $(S_i(1), S_i(0)) = (1, 0)$
  - non-compliers $\begin{cases} \textit{always}-\textit{takers} & (S_i(1), S_i(0)) = (1, 1) \\ \textit{never}-\textit{takers} & (S_i(1), S_i(0)) = (0, 0) \\ \textit{defiers} & (S_i(1), S_i(0)) = (0, 1) \end{cases}$
- Observed strata and compliance behavior:

	$Z_i = 1$	$Z_i = 0$
$S_i = 1$	Complier/Always-taker	Defier/Always-taker
$S_i = 0$	Defier/Never-taker	Complier/Never-taker

Principal causal effects

- Four principal causal effects
  - complier average causal effect (CACE): $\mathbb{E}\{Y_i(1) - Y_i(0) \mid U_i = (1, 0)\}$
  - always takers: $\mathbb{E}\{Y_i(1) - Y_i(0) \mid U_i = (1, 1)\}$
  - never takers: $\mathbb{E}\{Y_i(1) - Y_i(0) \mid U_i = (0, 0)\}$
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Principal causal effects

- Four principal causal effects
  - complier average causal effect (CACE): $\mathbb{E}\{Y_i(1) - Y_i(0) \mid U_i = (1,0)\}$
  - always takers: $\mathbb{E}\{Y_i(1) - Y_i(0) \mid U_i = (1,1)\}$
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  - defiers: $\mathbb{E}\{Y_i(1) - Y_i(0) \mid U_i = (0,1)\}$

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- CACE $\neq$ overall treatment effect unless the treatment effect for non-compliers equals CACE

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Outline

- Recap: potential outcomes and ACE
- Post-treatment variable
- Principal stratification
  - noncompliance
  - truncation by death
  - surrogate evaluation
- Identification and statistical inference
  - binary instrumental variable model
  - partial identification
  - principal ignorability
  - auxiliary independence
- Recent applications of principal stratification

Truncation by death

- Example 1: the effect of treatment (Z) on quality of life (Y) $\rightsquigarrow$ some patients may die before the outcome is observed
- Example 2: the effect of job training on hourly wage $\rightsquigarrow$ some subjects may be unemployed
- In example 1, S_i is the survival status and in example 2, S_i is the employment status

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- Traditional method treats the truncation by death problem as a standard missing data problem (censoring by death)
  - Heckman selection model: models for $\mathbb{E}(Y_i \mid Z_i, \mathbf{X}_i)$ and $\mathbb{E}(S_i \mid \mathbf{X}_i)$
  - assumes that the outcome is well-defined for all units

Principal stratification for truncation by death problem

- Four principal strata:
  - always survivors: $(S_i(1) = 1, S_i(0) = 1) \rightsquigarrow Y_i$ always well defined
  - $(S_i(1) = 1, S_i(0) = 0) \rightsquigarrow Y_i$ not well defined when $Z_i = 0$
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- Other principal causal effects are not well defined

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- Examples of surrogates
  - CD4 cell count as surrogate for the survival status of HIV patients
  - short-term survival as surrogate for long-term survival
  - kindergarten test scores as surrogate for long-run return

Catastrophic consequence using an invalid surrogate

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  - it was hypothesized that suppression of ventricular arrhythmias would reduce the rate of death
- Drugs found to suppress arrhythmia were approved by the FDA
  - more than 200 000 persons per year took these drugs
- Follow up trials showed that the drugs increased mortality
  - tens of thousands of patients died in America's worst drug disaster
  - the casualties were estimated to approach levels close to those of the war in Vietnam

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  - $\mathbb{E}\{h(S_i(1)) - h(S_i(0))\} = 0$ implies $\mathbb{E}\{Y_i(1) - Y_i(0)\} = 0$
  - there exist units with no causal effect of treatment on the surrogate but experience causal effects of treatment on outcome: $S_i(1) = S_i(0)$ but $Y_i(1) \neq Y_i(0)$

Properties of a good surrogate

- Intuitively, a good surrogate should satisfy:
  - if the treatment does not affect surrogate, it does not affect outcome
  - if the treatment affects surrogate, it affects outcome

- Formally, a good surrogate should satisfy:
  - causal necessity (Frangakis and Rubin 2002)

$$S_i(1) = S_i(0) \Longrightarrow Y_i(1) = Y_i(0)$$

  - causal sufficiency (Gilbert and Hudgens 2008)

$$S_i(1) \neq S_i(0) \Longrightarrow Y_i(1) \neq Y_i(0)$$

Principal stratification and principal causal effects

- A simple example: treatment comparison for AIDS patients, where CD4 is the candidate surrogate
- Four principal strata defined by $\{S_i(1), S_i(0)\}$

$$U_i \equiv \{S_i(1), S_i(0)\} \in \{HH, HL, LH, LL\}$$

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- Criteria for a principal surrogate
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  - causal sufficiency: $\mathrm{ACE}_{HL} \neq 0$ and $\mathrm{ACE}_{LH} \neq 0$
- Lauritzen (2004) proposes the criteria of a strong surrogate, which is stronger than the criteria of a principal surrogate

Statistical surrogate vs. principal surrogate

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- Evaluation
    - statistical surrogate: direct estimation from the observed data
    - principal surrogate: requires the identification and estimation of principal causal effects

Can we avoid the disaster using these criteria?

$$\begin{tikzcd} & & U \arrow[dl] \arrow[dr] & \\ Z \arrow[r, "+"] & S \arrow[rr, "+"] & & Y \end{tikzcd}$$

- Surrogate "paradox" (Chen, Geng and Jia 2007, Ju and Geng 2010)

- For a principal or statistical surrogate, it is still possible that
  - Positive causal effect of Z on S: $\text{ACE}(Z \to S) > 0$
  - Positive causal effect of S on Y: $\text{ACE}(S \to Y) > 0$ and $\text{Cor}(S, Y) > 0$
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- It is possible that drugs are beneficial for VA but increase mortality

Principal surrogates may suffer from surrogate paradox

- Proportions of principal strata, $\pi_u = \mathbb{P}(U_i = u)$:
  $(\pi_{HH}, \pi_{HL}, \pi_{LH}, \pi_{LL}) = (0.2, 0.4, 0.2, 0.2)$

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\begin{aligned}
& \mathsf{ACE}(Z \to S) = \mathbb{P}\{S_i(1) = 1\} - \mathbb{P}\{S_i(0) = 1\} \\
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- However, the ACE on the outcome is negative

$$
\mathsf{ACE}(Z \to Y) = \sum_u \pi_u \mathsf{ACE}_u = 0.4 \times 0.1 + 0.2 \times (-0.3) = -0.02 < 0
$$

New criteria: when a surrogate satisfies causal necessity

- $\mathsf{ACE}(Z \to S) > 0$ should imply $\mathsf{ACE}(Z \to Y) > 0$

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Surrogate evaluation in colon clinical trials (Jiang et al., 2016)

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  - $\text{ACE}_{00} = -0.001$ (s.e.$= 0.042$) and $\text{ACE}_{11} = 0.015$ (s.e.$= 0.012$)
  - $\text{ACE}_{10} = 0.774$ (s.e.$= 0.037$) and $\text{ACE}_{01} = -0.750$ (s.e.$= 0.054$)

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- DFS is a good surrogate

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Mediation analysis and principal stratification

- Principal strata direct effect: $\mathbb{E}\{Y(1) - Y(0) \mid S(1) = S(0) = s\}$
  - for units with $S(1) = S(0) = s$,
    $Y(1) - Y(0) = Y(1, S(1)) - Y(0, S(0)) = Y(1, S(1)) - Y(0, S(1))$
  - Principal strata direct effect is the natural direct effect within principal strata

- Principal strata indirect effect: $\mathbb{E}\{Y(1) - Y(0) \mid S(1) = 1, S(0) = 0\}$?

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  - $\mathbb{E}\{Y(1) - Y(0) \mid S(1) = 1, S(0) = 0\}$ consists of both direct and indirect effects

Outline

- Potential outcomes and ACE
- Post-treatment variable
- Principal stratification
  - noncompliance
  - truncation by death
  - surrogate evaluation
- Identification and statistical inference
  - binary instrumental variable model
  - partial identification
  - principal ignorability
  - auxiliary independence
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  - difficulty: $S_i(1)$ and $S_i(0)$ are not simultaneous observed $\rightsquigarrow$ U_i is latent

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Noncompliance

- Four principal strata (latent types):
  - compliers $(S_i(1), S_i(0)) = (1, 0)$,
  - non-compliers $\begin{cases} \textit{always} - \textit{takers} & (S_i(1), S_i(0)) = (1, 1), \\ \textit{never} - \textit{takers} & (S_i(1), S_i(0)) = (0, 0), \\ \textit{defiers} & (S_i(1), S_i(0)) = (0, 1) \end{cases}$
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- Observed strata and compliance behavior:

	$Z_i = 1$	$Z_i = 0$
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- $\text{CACE} = \text{ACE}_c = \mathbb{E}\{Y_i(1) - Y_i(0) \mid U_i = c\}$

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- $\text{ACE}(Z \to S) = \mathbb{E}\{S_i(1) - S_i(0)\} > 0 \rightsquigarrow$ there exists compliers
    - $\mathbb{E}\{S_i(1) - S_i(0)\} = \mathbb{P}\{S_i(1) = 1\} - \mathbb{P}\{S_i(0) = 1\} = \mathbb{P}\{S_i(1) = 1, S_i(0) = 0\}$

Monotonicity: implications

- Observed strata and compliance behavior under monotonicity

	$Z_i = 1$	$Z_i = 0$
$S_i = 1$	Complier/Always-taker	Always-taker
$S_i = 0$	Never-taker	Complier/Never-taker

- $\mathbb{P}(\text{Never-taker}) = \mathbb{P}\{S_i(1) = 0\} = \mathbb{P}(S_i = 0 \mid Z_i = 1)$
  $\mathbb{P}(\text{always-taker}) = \mathbb{P}\{S_i(0) = 1\} = \mathbb{P}(S_i = 1 \mid Z_i = 0)$
  $\mathbb{P}(\text{complier}) = \mathbb{P}(S_i = 1 \mid Z_i = 1) - \mathbb{P}(S_i = 1 \mid Z_i = 0)$

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    $Y_i(1) = Y_i(1, S_i(1)) = Y_i(1, S_i(0)) = Y_i(0, S_i(0)) = Y_i(0)$
    $\text{ACE}_a = \text{ACE}_n = 0$
  - compliers: $\text{ACE}_c = \mathbb{E}\{Y_i(1) - Y_i(0) \mid U_i = c\}$, $Z_i = S_i$

Identification

- ITT effect decomposition:

$$\begin{aligned} \text{ACE} &= \text{ACE}_c \times \Pr(\text{compliers}) + \text{ACE}_a \times \Pr(\text{always} - \text{takers}) \\ &\quad + \text{ACE}_n \times \Pr(\text{never} - \text{takers}) + \text{ACE}_d \times \Pr(\text{defiers}) \\ &= \text{ACE}_c \times \Pr(\text{compliers}) \end{aligned}$$

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$$
\begin{aligned}
\text{ACE}_c &= \frac{\text{ACE}(Z \to Y)}{\mathbb{P}(\text{compliers})} \\
&= \frac{\text{ACE}(Z \to Y)}{\text{ACE}(Z \to S)} \\
&= \frac{\mathbb{E}(Y_i \mid Z_i = 1) - \mathbb{E}(Y_i \mid Z_i = 0)}{\mathbb{E}(S_i \mid Z_i = 1) - \mathbb{E}(S_i \mid Z_i = 0)}
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Complier average causal effect

- CACE is identified by the ratio of two ITT effects
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- Encouragement design: randomly encourage people to take the treatment
  - CACE is the effect of people who would take the treatment only if encouraged
  - different encouragement yields different compliers

Estimation and inference

- Wald estimator: $\widehat{\text{CACE}} = \frac{\widehat{\mathbb{E}}(Y_i|Z_i=1)-\widehat{\mathbb{E}}(Y_i|Z_i=0)}{\widehat{\mathbb{E}}(S_i|Z_i=1)-\widehat{\mathbb{E}}(S_i|Z_i=0)}$

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- Observational studies: Abadies' Kappa

Evaluation of job training program

- A randomized field experiment investigating the efficacy of a job training intervention on unemployed workers
- Encouragement: Z_i; participation: S_i; job-search self-efficacy: Y_i

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- CACE: effect of participation on job-search self-efficacy for people who would participate if and only if they are encouraged
- ACE($Z \rightarrow Y$): est. $= 0.067$, s.e. $= 0.050$, 95% CI $= [-0.031,\ 0.166]$
- CACE: est. $= 0.109$, s.e. $= 0.081$, 95% CI $= [-0.050,\ 0.268]$

Effect of veteran status on earnings (Angrist, 1990, *AER*)

- There were five draft lotteries during the Vietnam War period. In each lottery, priority for induction was determined by a Random Sequence Number (RSN) from 1-365 that was assigned to birthdates in the cohort being drafted
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- Draft lottery RSNs were randomly assigned in a televised drawing held a few months before men reaching draft age were to be called
- Draft-eligibility ceilings were announced later in the year, once Defense Department manpower needs were known
- Subsequent selection from the draft-eligible pool was based on a number of criteria: physical examination and a mental aptitude test

Setup

- Z_i: draft-eligibility; S_i: veteran status; outcome Y_i earnings in 1981-1984

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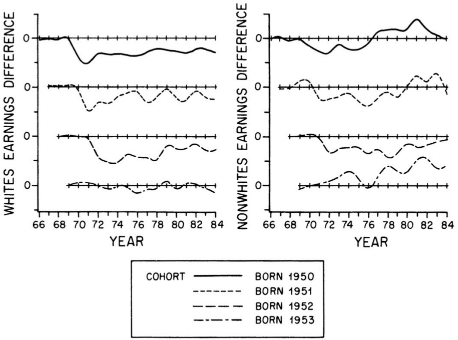

Notes: The figure plots the difference in FICA taxable earnings by draft-eligibility status for the four cohorts born 1950–53. Each tick on the vertical axis represents $500 real (1978) dollars.

FIGURE 2. THE DIFFERENCE IN EARNINGS BY DRAFT-ELIGIBILITY STATUS

Wald estimates

Table 3—Wald Estimates

Cohort	Year	Draft-Eligibility Effects in Current $: FICA Earnings (1)	Adjusted FICA Earnings (2)	Total W-2 Earnings (3)	$\hat{p}^e - \hat{p}^n$ (4)	Service Effect in 1978 $ (5)
1950	1981	−435.8	−487.8	−589.6	0.159	−2,195.8
		(210.5)	(237.6)	(299.4)	(0.040)	(1,069.5)
	1982	−320.2	−396.1	−305.5		−1,678.3
		(235.8)	(281.7)	(345.4)		(1,193.6)
	1983	−349.5	−450.1	−512.9		−1,795.6
		(261.6)	(302.0)	(441.2)		(1,204.8)
	1984	−484.3	−638.7	−1,143.3		−2,517.7
		(286.8)	(336.5)	(492.2)		(1,326.5)
1951	1981	−358.3	−428.7	−71.6	0.136	−2,261.3
		(203.6)	(224.5)	(423.4)	(0.043)	(1,184.2)
	1982	−117.3	−278.5	−72.7		−1,386.6
		(229.1)	(264.1)	(372.1)		(1,312.1)
	1983	−314.0	−452.2	−896.5		-2,181.8
		(253.2)	(289.2)	(426.3)		(1,395.3)
	1984	−398.4	−573.3	−809.1		−2,647.9
		(279.2)	(331.1)	(380.9)		(1,529.2)
1952	1981	−342.8	−392.6	−440.5	0.105	−2,502.3
		(206.8)	(228.6)	(265.0)	(0.050)	(1,556.7)
	1982	−235.1	−255.2	−514.7		−1,626.5
		(232.3)	(264.5)	(296.5)		(1,685.8)
	1983	−437.7	−500.0	−915.7		−3,103.5
		(257.5)	(294.7)	(395.2)		(1,829.2)
	1984	−436.0	−560.0	−767.2		−3,323.8
		(281.9)	(330.1)	(376.0)		(1,959.3)

Physical activity and weight following car ownership in Beijing, China: quasi-experimental cross sectional study

Michael L Anderson,[1] Fangwen Lu,[2] Jun Yang[3]

- In January 2011, to deal with the problem of congestion, Beijing capped the number of new vehicles allowed at 240000 each year and introduced a vehicle permit (license plate) lottery
- After that date, only residents who entered and won the lottery were entitled to a license plate.
- The lottery was drawn monthly, and winners had to purchase a car within six months of winning. By mid-2012 the probability of winning fell below 2% a month

Effect of winning the lottery

- Z_i: winning the lottery; S_i: buying a car; Y_i: weekly transit rides, minute daily walking/bicycling, weight

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Table 3 | Age stratified associations between winning the lottery and transit use, activity, and weight

	Time since winning (95% CI)		
Dependent variables	**0.1 years (minimum)**	**2.6 years (average)**	**5.1 years (maximum)**
Individuals aged ≥40			
Weekly transit rides	−2.18 (−4.13 to −0.24)	−2.1 (−3.35 to −0.85)	−2.02 (−5.16 to 1.12)
Minutes daily walking/bicycling	12.1 (−4.66 to 28.86)	−2.59 (−12.12 to 6.94)	−17.29 (−36.52 to 1.95)
Weight (kg)	1.29 (−5.07 to 7.65)	3.24 (−0.31 to 6.8)	5.2 (−2.59 to 12.99)
Individuals aged ≥50			
Weekly transit rides	−2.88 (−5.57 to −0.19)	−1.9 (−3.61 to −0.18)	−0.91 (−5.45 to 3.63)
Minutes daily walking/bicycling	27.4 (−0.28 to 55.08)	−1.19 (−13.76 to 11.38)	−29.78 (−54.08 to −5.49)
Weight (kg)	−1 (−8.4 to 6.4)	4.67 (0.04 to 9.31)	10.34 (0.49 to 20.19)

Effect of car ownership

- Ratio estimation for the effect of buying a car
- High compliance rate: 0.91 $(0.89, 0.94)$ $\rightsquigarrow$ effect of winning the lottery $\approx$ effect of buying a car

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- Limitation
  - weight is self-reported
  - target population: people who want to and are able to buy a car

Outline

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- Exclusion restriction implies $\text{SACE} = 0 \rightsquigarrow$ cannot be invoked
  - without exclusion restriction, SACE is not identifiable
  - alternative identification assumptions?
- Partial identification $\rightsquigarrow$ bounds on SACE
  - find all the possible values of SACE that are compatible with the observed data

Bounds for survivor average causal effect

- Randomization and monotonicity hold
- Proportions of principal strata:
  $\pi_{00} = \mathbb{P}(S_i = 0 \mid Z_i = 1)$, $\pi_{11} = \mathbb{P}(S_i = 1 \mid Z_i = 0)$, and
  $\pi_{10} = 1 - \pi_{00} - \pi_{11} = \mathbb{P}(S_i = 1 \mid Z_i = 1) - \mathbb{P}(S_i = 1 \mid Z_i = 0)$

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- Key: bounds on $\mathbb{E}\{Y_i(1) \mid S_i(1) = 1, S_i(0) = 1\}$

Bounds on survivor average causal effect

- The observed stratum $(Z_i = 1, S_i = 1)$ is a mixture of two strata $U_i = 11$ and $U_i = 10$

$$
\begin{aligned}
& \mathbb{E}(Y_i = 1 \mid Z_i = 1, S_i = 1) = \mathbb{E}\{Y_i(1) = 1 \mid U = 11/10\} \\
= \quad & \mathbb{E}\{Y_i(1) \mid U = 11\}\mathbb{P}(U = 11 \mid U = 11/10) \\
& + \mathbb{E}\{Y_i(1) \mid U = 10\}\mathbb{P}(U = 10 \mid U = 11/10) \\
= \quad & \frac{\pi_{11}}{\pi_{11} + \pi_{10}} \mathbb{E}\{Y_i(1) \mid S_i(1) = 1, S_i(0) = 1\} \\
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- Y is bounded in $[l, u]$ $\rightsquigarrow$ bounds on $\mathbb{E}\{Y_i(1) \mid S_i(1) = 1, S_i(0) = 1\}$

$$
\begin{aligned}
\text{Upper} &= \frac{(\pi_{11} + \pi_{10})\mathbb{E}(Y_i = 1 \mid Z_i = 1, S_i = 1) - l \cdot \pi_{10}}{\pi_{11}} \\
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- Statistical inference is hard; even harder without monotonicity

Acute respiratory distress syndrome network study

$Z = 1$				$Z = 0$			
	$Y = 1$	$Y = 0$	total		$Y = 1$	$Y = 0$	total
$S = 1$	54	268	302	$S = 1$	59	218	277
$S = 0$	*	*	109	$S = 0$	*	*	152

- 861 patients with lung injury and acute respiratory distress syndrome were randomized to receive mechanical ventilation with either lower tidal volumes ($Z_i = 1$) or traditional tidal volumes ($Z_i = 0$)
- Outcome: breathe without assistance by day 28 (1 for not)

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- Proportions of principal strata

$$\widehat{\pi}_{11} = \frac{277}{277+152} = 0.646, \quad \widehat{\pi}_{00} = \frac{109}{109+302} = 0.265, \quad \widehat{\pi}_{10} = 0.089$$

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- Sample means of the outcome for survived patients

$$\widehat{\mathbb{E}}(Y_i \mid Z_i = 1, M_i = 1) = \frac{54}{302} = 0.179, \quad \widehat{\mathbb{E}}(Y_i \mid Z_i = 0, M_i = 1) = \frac{59}{277} = 0.213$$

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- Bounds on $\mathbb{E}\{Y_i(1) \mid S_i(1) = 1, S_i(0) = 1\}$: $[0.065, 0.203]$
- Bounds on SACE: $[-0.147, -0.010]$

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- Principal ignorability $Y_i(z) \perp\!\!\!\perp U_i \mid (Z_i = z, S_i = s, \mathbf{X}_i)$
  - expected potential outcome is the same across different principal strata within each observed stratum
  - implies $\mathbb{E}\{Y_i(1) \mid U_i = 11, Z_i = 1, S_i = 1, \mathbf{X}_i\} = \mathbb{E}\{Y_i(1) \mid U_i = 10, Z_i = 1, S_i = 1, \mathbf{X}_i\}$

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    $\mathbb{P}(U_i = 11 \mid \mathbf{X}_i) = \mathbb{P}(S_i = 1 \mid Z_i = 0, \mathbf{X}_i)$,
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  - $\mathbb{E}\{Y_i(1) \mid U_i = 10, Z_i = 1, S_i = 1, \mathbf{X}_i\} = \mathbb{E}\{Y_i(1) \mid Z_i = 1, S_i = 1, \mathbf{X}_i\}$

Weighting method

Theorem

$$
\begin{aligned}
\mathrm{ACE}_{10} &= \mathbb{E}\{w_{1,10}(\mathbf{X})Y \mid Z=1, S=1\} - \mathbb{E}\{w_{0,10}(\mathbf{X})Y \mid Z=0, S=0\} \\
\mathrm{ACE}_{00} &= \mathbb{E}\{Y \mid Z=1, S=0\} - \mathbb{E}\{w_{0,00}(\mathbf{X})Y \mid Z=0, S=0\} \\
\mathrm{ACE}_{11} &= \mathbb{E}\{w_{1,11}(\mathbf{X})Y \mid Z=1, S=1\} - \mathbb{E}\{Y \mid Z=0, S=1\}
\end{aligned}
$$

$$e_u(\mathbf{X}) = \mathbb{P}(U_i = u \mid \mathbf{X}) \quad e_u = \mathbb{E}\{e_u(\mathbf{X})\}$$

$$w_{1,10} = \frac{e_{10}(\mathbf{X})}{e_{10}(\mathbf{X}) + e_{11}(\mathbf{X})} \Bigg/ \frac{e_{10}}{e_{10} + e_{11}} \quad w_{0,11} = \frac{e_{10}(\mathbf{X})}{e_{10}(\mathbf{X}) + e_{00}(\mathbf{X})} \Bigg/ \frac{e_{10}}{e_{10} + e_{00}}$$

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Estimation and inference

- Estimation steps
  1. estimate the principal score $\mathbb{P}(U_i \mid \mathbf{X}_i)$, e.g., logistic model
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- A unification in observational studies (Jiang et al., 2022)
  - treatment ignorability, principal ignorability, monotonicity
  - triple robustness: propensity score, principal score, outcome model

Comparison of different identification strategies

- A1: $Z_i \perp\!\!\!\perp \{Y_i(1), Y_i(0), S_i(1), S_i(0)\}$;
  A2: $Z_i \perp\!\!\!\perp \{Y_i(1), Y_i(0), S_i(1), S_i(0)\} \mid \mathbf{X}_i$
- B1: monotonicity and exclusion restriction;
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- Other strategies: additional information (e.g., auxiliary independence), likelihood based inference, Bayesian analysis

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- Identification with multiple trials in Jiang et al. (2016)
  - treatment Z_i; three-year cancer reoccurrence S_i; five-year survival Y_i
  - trial number W_i: potential survival status does not depend on the trial number given physical status

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Identification with discrete S_i

- $S \in \{s_1, \ldots, s_K\}$ and $W \in \{w_1, \ldots, w_L\}$
- M_{s_0}: $K \times L$ matrix with (k, l)-th element $\mathbb{P}(S_1 = s_k \mid S_0 = s_0, W = w_l)$
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- $\mathbb{P}(Y_0 \mid S_1, S_0 = s_0)$ is identifiable if $\mathrm{rank}(M_{s_0}^\top M_{s_0}) = K$
- $\mathbb{P}(Y_1 \mid S_1 = s_1, S_0)$ is identifiable if $\mathrm{rank}(M_{s_1}^\top M_{s_1}) = K$

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- Binary S_i and W_i

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- Solve $\mathbb{E}(Y_0 \mid S_1)$ from equations

$$
\begin{aligned}
\delta_1 &= \mathbb{E}(Y_0 \mid S_1 = 1)\theta_{11} + \mathbb{E}(Y_0 \mid S_1 = 0)\theta_{01} \\
\delta_0 &= \mathbb{E}(Y_0 \mid S_1 = 1)\theta_{10} + \mathbb{E}(Y_0 \mid S_1 = 0)\theta_{00}
\end{aligned}
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- General case in Jiang and Ding (2021)

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Principal stratification based on outcomes

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Principal stratification based on outcomes

- Principal stratification does not require S to be on the causal pathway from Z to Y
- Examples of principal stratification based on outcomes
  - proportion of units benefit from treatment: $\mathbb{P}\{Y(1) > Y(0)\}$
  - probability of necessity: $\mathbb{P}\{Y(0) = 0 \mid Z = 1, Y = 1\}$;
    probability of sufficiency: $\mathbb{P}\{Y(1) = 1 \mid Z = 0, Y = 0\}$

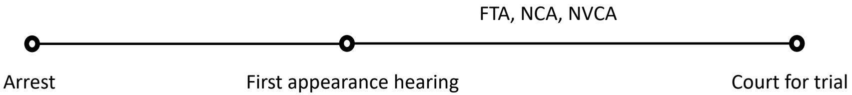
Principal stratification based on outcomes

- Principal stratification does not require S to be on the causal pathway from Z to Y

- Examples of principal stratification based on outcomes
  - proportion of units benefit from treatment: $\mathbb{P}\{Y(1) > Y(0)\}$
  - probability of necessity: $\mathbb{P}\{Y(0) = 0 \mid Z = 1, Y = 1\}$;
    probability of sufficiency: $\mathbb{P}\{Y(1) = 1 \mid Z = 0, Y = 0\}$

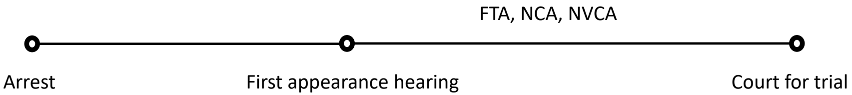
- Evaluation of algorithm-assisted human decision making
  - consequential decisions made by judges, doctors, etc.
  - pre-trial risk assessment instrument

First appearance hearings



- Judges decide pre-trial release conditions
  - bail and monitoring
  - many cases in one day

First appearance hearings



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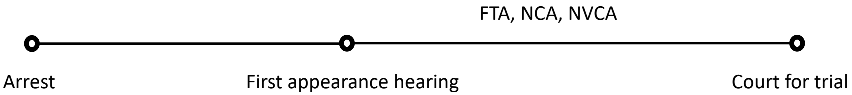

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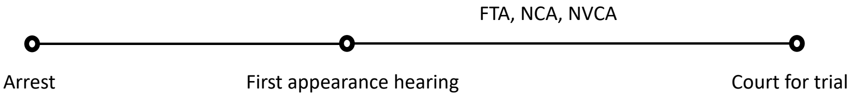

- Judges decide pre-trial release conditions
  - bail and monitoring
  - many cases in one day
- Presumption of innocence: judges balance between
  - cost of pre-trial detention
  - risk of arrestee
- Judges are required to consider three negative outcomes
  1. arrestee may fail to appear in trial court (FTA)
  2. arrestee may engage in new criminal activity (NCA)
  3. arrestee may engage in new violent criminal activity (NVCA)

Pretrial Public Safety Assessment (PSA)

- PSA as an algorithmic recommendation

- PSA scores
  1. calculated based on nine factors
  2. two 6-point scores for FTA and NCA
  3. one binary score for NVCA

Failure to Appear: Points

PSA FACTOR	RESPONSE	POINTS
Pending charge at the time of the arrest	No	0
	Yes	1
Prior conviction (misdemeanor or felony)	No	0
	Yes	1
Prior failure to appear in the past 2 years	No	0
	Yes, just 1	2
	Yes, 2 or more	4
Prior failure to appear older than 2 years	No	0
	Yes	1

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- PSA as an algorithmic recommendation

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  1. calculated based on nine factors
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- Decision Making Framework (DMF)
  - combines scores for bail recommendation:
    cash bail or signature bond

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DANE COUNTY CLERK OF COURTS
Public Safety Assessment – Report

215 S Hamilton St #1000
Madison, WI 53703
Phone: (608) 266-4311

Name: [redacted]
Spillman Name Number: [redacted]
DOB: [redacted]
Gender: Male
Arrest Date: 03/25/2017
PSA Completion Date: 03/27/2017

New Violent Criminal Activity Flag

No

New Criminal Activity Scale

1	2	3	4	5	6

Failure to Appear Scale

1	2	3	4	5	6

Charge(s):

961.41(1)(D)(1) MFC DELIVER HEROIN <3 GMS F 3

Risk Factors:	**Responses:**
1. Age at Current Arrest	23 or Older
2. Current Violent Offense	No
a. Current Violent Offense & 20 Years Old or Younger	No
3. Pending Charge at the Time of the Offense	No
4. Prior Misdemeanor Conviction	Yes
5. Prior Felony Conviction	Yes
a. Prior Conviction	Yes
6. Prior Violent Conviction	2
7. Prior Failure to Appear Pretrial in Past 2 Years	0
8. Prior Failure to Appear Pretrial Older than 2 Years	Yes
9. Prior Sentence to Incarceration	Yes

Recommendations:

Release Recommendation - Signature bond

Conditions - Report to and comply with pretrial supervision

A field experiment for evaluating the PSA

- A field experiment in Dane county, Wisconsin
  - PSA is generated for each case using a computer system
  - randomly make PSA reports available to judges
  - mid-2017 – 2019 (randomization), 2-year follow-up

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  1. signature bond
  2. $\leq$ \$1,000 cash bond (small)
  3. $>$ \$1,000 cash bond (large)

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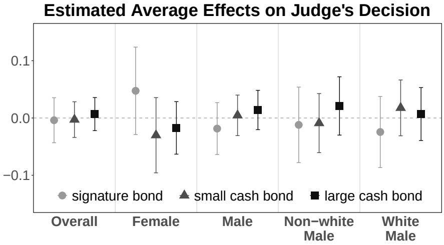
- Z_i: PSA provision indicator
  S_i: judge's decision (0 for signature bond, 1 for small cash, and 2 for large cash)
  Y_i: FTA, NCA, or NVCA

Intention-to-treat analysis of PSA provision



- ACE($Z \to S$) and ACE($Z \to Y$) $\rightsquigarrow$ overall effects of PSA provision

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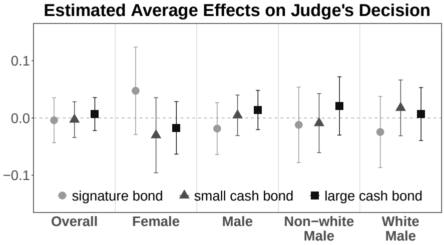

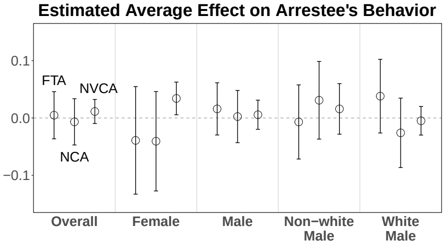

- ACE$(Z \rightarrow S)$ and ACE$(Z \rightarrow Y)$ $\rightsquigarrow$ overall effects of PSA provision
  - insignificant effects on judges' decisions
  - possible effect on NVCA for females

Issue of ITT analysis

- ITT effect on the negative outcomes
    - answers whether PSA provision helps prevent FTA, NCA, and NVCA

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- ITT effect on the judge's decision
  - answers whether PSA provision makes decision harsher or more lenient
  - does not answer whether PSA provision helps make better decisions
- Good decisions: detain risky arrestees and release safe arrestees

The meaning of a "good" decision varies

- A toy example

	if released	if detained
Arrestee A	NCA	no NCA
Arrestee B	no NCA	no NCA
Arrestee C	NCA	NCA

	without PSA	with PSA
Judge 1	release all	detain A,C
Judge 2	release all	detain B,C

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  - ITT effects of PSA are the same
  - Risk level: C $>$ A $>$ B
    - detaining A prevents an NCA
    - detaining B is unnecessary
    - detaining C does not help

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- Risk levels depend on $\left\{ \begin{array}{l} \text{(potential) outcome if released} \\ \text{(potential) outcome if detained} \end{array} \right.$

Setup

- Notation
  - Z_i: PSA provision indicator
  - S_i: 1 for detention, 0 for release
  - Y_i: binary outcome (e.g., NCA)
  - X_i: observed covariates
  - U_i: unobserved covariates

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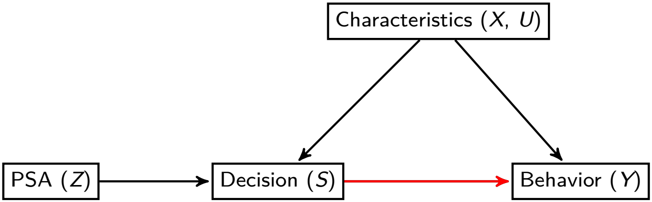
- Potential outcomes
  - $D_i(z)$: potential value of the decision when $Z_i = z$
  - $Y_i(z, s)$: potential outcome when $Z_i = z$ and $S_i = s$
  - No interference across cases: first arrests only

Assumptions



- Randomized treatment assignment: $\{S_i(z), Y_i(z, s), X_i, U_i\} \perp\!\!\!\perp Z_i$

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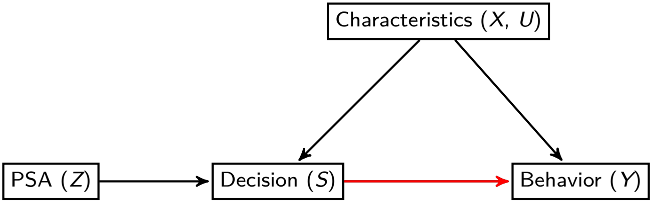

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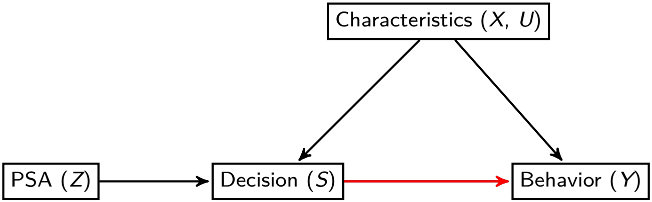

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Defining risk levels based on principal stratification

- Principal stratification
  - $(Y_i(1), Y_i(0)) = (0, 1)$: preventable cases
  - $(Y_i(1), Y_i(0)) = (1, 1)$: risky cases
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  - $\cancel{(Y_i(1), Y_i(0)) = (1, 0)}$: eliminated by monotonicity

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- Average principal causal effects of PSA on judges' decisions:

$$
\begin{aligned}
\text{APCEp} &= \mathbb{E}\{S_i(1) - S_i(0) \mid Y_i(1) = 0, Y_i(0) = 1\}, \\
\text{APCEr} &= \mathbb{E}\{S_i(1) - S_i(0) \mid Y_i(1) = 1, Y_i(0) = 1\}, \\
\text{APCEs} &= \mathbb{E}\{S_i(1) - S_i(0) \mid Y_i(1) = 0, Y_i(0) = 0\}.
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\end{aligned}
$$

- If PSA is helpful, we should have APCEp > 0 and APCEs < 0.
- The desirable sign of APCEr depends on various factors.

Partial identification

$$
\begin{aligned}
\text{APCEp} &= \frac{\mathbb{P}(Y_i = 1 \mid Z_i = 0) - \mathbb{P}(Y_i = 1 \mid Z_i = 1)}{\mathbb{P}\{Y_i(0) = 1\} - \mathbb{P}\{Y_i(1) = 1\}} \\
\text{APCEr} &= \frac{\mathbb{P}(S_i = 1, Y_i = 1 \mid Z_i = 1) - \mathbb{P}(S_i = 1, Y_i = 1 \mid Z_i = 0)}{\mathbb{P}\{Y_i(1) = 1\}} \\
\text{APCEs} &= \frac{\mathbb{P}(S_i = 0, Y_i = 0 \mid Z_i = 0) - \mathbb{P}(S_i = 0, Y_i = 0 \mid Z_i = 1)}{1 - \mathbb{P}\{Y_i(0) = 1\}}
\end{aligned}
$$

- The signs of principal causal effects are identifiable under the assumptions of randomization, exclusion restriction, and monotonicity.

Extension to Ordinal Decision

- Judges decisions are typically ordinal (e.g., bail amount)
  - $S_i = 0, 1, \ldots, k$: a bail of increasing amount
  - Monotonicity: $Y_i(s_1) \;\geq\; Y_i(s_2)$ for $s_1 \leq s_2$

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  - Monotonicity: $Y_i(s_1) \;\geq\; Y_i(s_2)$ for $s_1 \leq s_2$
- Principal strata based on an ordinal measure of risk

$$R_i \;=\; \begin{cases} \min\{s : Y_i(s) = 0\} & \text{if } Y_i(k) = 0 \\ k + 1 & \text{if } Y_i(k) = 1 \end{cases}$$

  - Least amount of bail that keeps an arrestee from committing NCA

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- Principal strata based on an ordinal measure of risk

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  - Least amount of bail that keeps an arrestee from committing NCA
  - Example with $k = 2$

principal strata	$(Y_i(0), Y_i(1), Y_i(2))$	R_i
risky cases	$(1, 1, 1)$	3
preventable cases	$(1, 1, 0)$	2
easily preventable cases	$(1, 0, 0)$	1
safe cases	$(0, 0, 0)$	0

APCE for ordinal decision

- For people with $R_i = r$
  - judges make decision $S_i \geq r \rightsquigarrow$ not commit a crime
  - judges make decision $S_i < r \rightsquigarrow$ commit a crime

- Causal quantities of interest : reduction in the proportion of NCA attributable to PSA provision

$$\mathsf{ACEP}(r) \;=\; \Pr\{S_i(1) \geq r \mid R_i = r\} - \Pr\{S_i(0) \geq r \mid R_i = r\}$$

- Partial identification without unconfoundedness

Point identification

- Unconfoundedness: $Y_i(s) \perp\!\!\!\perp S_i \mid \mathbf{X}_i, Z_i = z$
- Violation of unconfoundedness
  - unobserved covariates between decision and outcome
  - sensitivity analysis

Point identification

- Unconfoundedness: $Y_i(s) \perp\!\!\!\perp S_i \mid \mathbf{X}_i, Z_i = z$
- Violation of unconfoundedness
  - unobserved covariates between decision and outcome
  - sensitivity analysis
- Principal score

$$e_r(\mathbf{x}) = \Pr(R_i = r \mid \mathbf{X}_i = \mathbf{x})$$

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- Unconfoundedness: $Y_i(s) \perp\!\!\!\perp S_i \mid \mathbf{X}_i, Z_i = z$
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  - sensitivity analysis
- Principal score

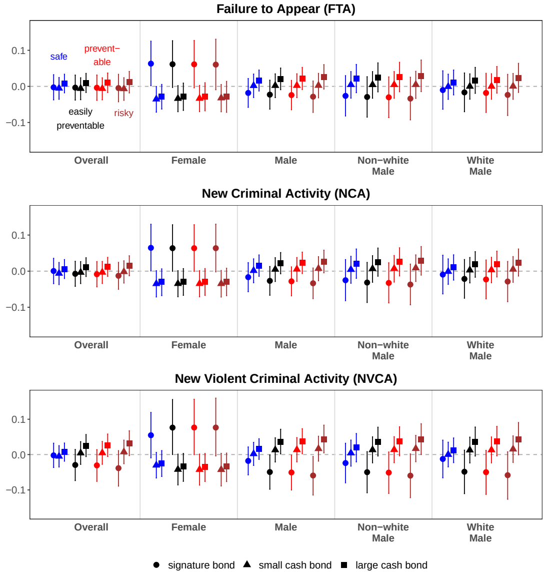
$$e_r(\mathbf{x}) \quad = \quad \Pr(R_i = r \mid \mathbf{X}_i = \mathbf{x})$$

- Identification formula

$$\mathsf{ACEP}(r) \;=\; \mathbb{E}\left[\underbrace{\frac{e_r(\mathbf{x})}{\mathbb{E}\{e_r(\mathbf{X}_i)\}}}_{\text{weight}} \mathbf{1}(S_i \geq r) \mid Z_i = 1\right] - \mathbb{E}\left[\underbrace{\frac{e_r(\mathbf{x})}{\mathbb{E}\{e_r(\mathbf{X}_i)\}}}_{\text{weight}} \mathbf{1}(S_i \geq r) \mid Z_i = 0\right]$$

Estimated average principal causal effects

$$\Pr\{S_i(1) = s \mid R_i = r\} - \Pr\{S_i(0) = s \mid R_i = r\}$$



Topics not covered

- Principal stratification in observational studies
  - identification assumptions
  - outcome regression, inverse probability weighting, multiply robust estimation
- Other identification strategies
  - parametric modeling
  - Bayesian analysis
  - using additional information, e.g., secondary outcome
- More complex settings
  - interference
  - data complications: missing data, selection bias, measurement error

Summary

- Post-treatment variable S_i: affected by the treatment
  - naive adjustment for the observed variable S_i does not yield a valid causal quantity
  - principal stratification defined by $S_i(1)$ and $S_i(0)$
  - application to non-compliance, truncation-by-death, and surrogate evaluation problems

- Various identification strategies
  - monotonicity and exclusion restriction
  - monotonicity and principal ignorability

- Extension
  - identification without monotonicity
  - discrete and continous S