$$E(\varphi) = \frac{1}{(n+1)V}\int_X \varphi \sum_{i=0}^{n} \omega^i \wedge \omega_\varphi^{n-i}.$$

Choose $f \in C^\infty(X, \mathbb{R})$.

$$\frac{d}{dt}\Big|_{t=0} E(\varphi+tf) = \frac{d}{dt}\Big|_{t=0}\left(\frac{1}{(n+1)V}\int_X (\varphi+tf)\sum_{i=0}^{n}\omega^i\wedge\omega_{\varphi+tf}^{n-i}\right)$$

$$= \frac{1}{(n+1)V}\int_X f\sum_{i=0}^{n}\omega^i\wedge\omega_\varphi^{n-i} + \frac{1}{(n+1)V}\int_X \varphi\sum_{i=0}^{n-1}(n-i)\,dd^c f\wedge\omega^i\wedge\omega_\varphi^{n-1-i}$$

$$= \frac{1}{(n+1)V}\int_X f\sum_{i=0}^{n}\omega^i\wedge\omega_\varphi^{n-i} + \frac{1}{(n+1)V}\int_X f\sum_{i=0}^{n-1}(n-i)(\omega_\varphi-\omega)\wedge\omega^i\wedge\omega_\varphi^{n-1-i}$$

$$= \frac{1}{(n+1)V}\left(\int_X f\sum_{i=0}^{n}\omega^i\wedge\omega_\varphi^{n-i} + \int_X f\sum_{i=0}^{n-1}(n-i)\left(\omega^i\wedge\omega_\varphi^{n-i} - \omega^{i+1}\wedge\omega_\varphi^{n-1-i}\right)\right)$$

$$= \frac{1}{(n+1)V}\left(\int_X f\sum_{i=0}^{n}\omega^i\wedge\omega_\varphi^{n-i} + (n+1)\int_X f\omega_\varphi^n - \int_X f\sum_{i=0}^{n}\omega^i\wedge\omega_\varphi^{n-i}\right)$$

$$= \frac{1}{V}\int_X f\,\omega_\varphi^n.$$

- Cor: For $\forall\, \varphi_0, \varphi_1 \in \mathcal{H}_\omega$ we have

$$\frac{1}{V}\int_X(\varphi_1-\varphi_0)\,\omega_{\varphi_1}^n \leq E(\varphi_1) - E(\varphi_0) \leq \frac{1}{V}\int_X(\varphi_1-\varphi_0)\,\omega_{\varphi_0}^n.$$

pf: Put $\varphi_t := (1-t)\varphi_0 + t\varphi_1$, $t\in[0,1]$.

$\varphi_t \in \mathcal{H}_\omega$. Put $f(t) := E(\varphi_t)$.

$$\frac{d}{dt}f(t) = \frac{1}{V}\int_X \dot\varphi_t\,\omega_{\varphi_t}^n = \frac{1}{V}\int_X(\varphi_1-\varphi_0)\,\omega_{\varphi_t}^n.$$

$$\frac{d^2}{dt^2}f(t) = \frac{n}{V}\int_X(\varphi_1-\varphi_0)\,dd^c\dot\varphi_t\wedge\omega_{\varphi_t}^{n-1} = \frac{n}{V}\int_X(\varphi_1-\varphi_0)\,dd^c(\varphi_1-\varphi_0)\wedge\omega_{\varphi_t}^{n-1}$$

$$= -\frac{n}{V}\int_X d(\varphi_1-\varphi_0)\wedge d^c(\varphi_1-\varphi_0)\wedge\omega_{\varphi_t}^{n-1} \leq 0.$$

$f''(t) \le 0 \Rightarrow f$ is concave.

Then $E(\varphi_1) - E(\varphi_0) = f(1) - f(0) = \frac{f(1)-f(0)}{1} \le f'(0)$

$f'(0) = \frac{1}{V}\int_X (\varphi_1 - \varphi_0)\,\omega_{\varphi_0}^n$. $\square$.

- Cor. For $\forall\, \varphi \in \mathcal{H}_\omega$, we have

① $\sup\varphi - E(\varphi) \ge \frac{1}{n}\left(E(\varphi) - \frac{1}{V}\int_X \varphi\,\omega_\varphi^n \right)$.

② $\sup\varphi - \frac{1}{V}\int_X \varphi\,\omega_\varphi^n \ge \frac{n+1}{n}\left(E(\varphi) - \frac{1}{V}\int_X \varphi\,\omega_\varphi^n \right)$.

Pf ①: Let $\varphi_t := t\varphi$. $t \in [0,1]$.

let $F(t) = \frac{1}{V}\int_X \varphi_t\,\omega^n - E(\varphi_t)$.

$G(t) = \frac{1}{n}\left(E(\varphi_t) - \frac{1}{V}\int_X \varphi_t\,\omega_{\varphi_t}^n \right)$.

$F(0) = G(0) = 0$ $\quad F(t) \le G(t)$.

Goal: $F'(t) \le G'(t)$.

$F'(t) = \frac{1}{V}\int_X \varphi\,\omega^n - \frac{1}{V}\int_X \varphi\,\omega_{t\varphi}^n$.

$G'(t) = \frac{1}{n}\frac{1}{V}\int_X \varphi\,\omega_{t\varphi}^n - \frac{1}{nV}\int_X \varphi\,\omega_{t\varphi}^n - \frac{1}{V}\int_X t\varphi\, dd^c\varphi \wedge \omega_{t\varphi}^{n-1}$

Observe: $-\frac{1}{V}\int_X t\varphi\, dd^c\varphi \wedge \omega_{t\varphi}^{n-1} = -\frac{1}{V}\int_X \varphi(\omega_{t\varphi} - \omega) \wedge \omega_{t\varphi}^{n-1}$

$= -\frac{1}{V}\int_X \varphi\,\omega_{t\varphi}^n + \frac{1}{V}\int_X \varphi\,\omega \wedge \omega_{t\varphi}^{n-1}$.

$G'(t) = \cancel{\frac{1}{n}\frac{1}{V}\int_X \varphi\,\omega_{t\varphi}^n - \frac{1}{nV}\int_X \varphi\,\omega_{t\varphi}^n} - \frac{1}{V}\int_X \varphi\,\omega_{t\varphi}^n + \frac{1}{V}\int_X \varphi\,\omega \wedge \omega_{t\varphi}^{n-1}$

Observe: $\frac{1}{V}\int_X \varphi(\omega^n - \omega \wedge \omega_{t\varphi}^{n-1}) \ge 0$

Indeed, $\frac{1}{V}\int_X \varphi(\omega^n - \omega\wedge\omega_{t\varphi}^{n-1})$

$a^{n-1}-b^{n-1}=(a-b)\sum_{i=0}^{n-2}a^i b^{n-2-i}$

$= \frac{1}{V}\int_X \varphi\, \omega\wedge(\omega^{n-1} - \omega_{t\varphi}^{n-1})$

$= \frac{1}{V}\int_X \varphi\, \omega\wedge(-dd^c t\varphi \wedge \sum_{i=0}^{n-2} \omega^i\wedge\omega_{t\varphi}^{n-2-i})$

$= \frac{1}{V}\int_X t\, \underline{d\varphi\wedge d^c\varphi\wedge\omega\wedge\sum_{i=0}^{n-2}\omega^i\wedge\omega_{t\varphi}^{n-2-i}} \geq C\sum\left|\frac{\partial\varphi}{\partial z_i}\right|^2 dz_i\ldots$

$p\in X$ $\quad \omega = \sum_{i=1}^n \sqrt{-1}\, dz^i\wedge d\bar z^i$.

$\Rightarrow F'(t) \leq G'(t)$. $\Rightarrow E(1)\leq G(1)$. $\square$.

$\omega_{t\varphi} = \sum_{i=1}^n \sqrt{-1}\lambda_i\, dz^i\wedge d\bar z^i$

$\lambda_i>0 \quad \lambda_i \geq C>0$.

$\omega_{t\varphi}\geq C\,\omega$.

- Prop: D is coercive iff $\delta^A > 1$.

"$\delta^A>1 \Rightarrow D$ is coercive".

$D(\varphi) = -\log\int_X e^{-\varphi}\,dV - E(\varphi)$

$a\in(0,1)$

$= -\log\int_X e^{-(a+(1-a))\varphi}\,dV - E(\varphi)$ $\quad p,q>0.\ \frac{1}{p}+\frac{1}{q}=1$.

$\geq -\frac{1}{p}\log\int_X e^{-ap\varphi}dV - \frac{1}{q}\log\int_X e^{-(1-a)q\varphi}dV - E(\varphi)$.

Choose $\delta\in(1,\delta^A([\omega]))$, $\alpha\in(0,\min\{1,\alpha([\omega])\})$.

let $\begin{cases} ap=\delta \\ (1-a)q=\alpha \end{cases} \Rightarrow \begin{cases} a=\frac{\delta(1-\alpha)}{\delta-\alpha}. \\ p=\frac{\delta-\alpha}{1-\alpha} \\ q=\frac{\delta-\alpha}{\delta-1}. \end{cases}$

$D(\varphi) \geq -\frac{1-\alpha}{\delta-\alpha}\log\int_X e^{-\delta\varphi}dV - \frac{\delta-1}{\delta-\alpha}\log\int_X e^{-\alpha\varphi}dV - E(\varphi)$.

$= -\frac{1-\alpha}{\delta-\alpha}\log\int_X e^{-\delta(\varphi-E(\varphi))}dV - \frac{\delta-1}{\delta-\alpha}\log\int_X e^{-\alpha(\varphi-\sup\varphi)}dV - E(\varphi)$

$+\delta\frac{1-\alpha}{\delta-\alpha}E(\varphi) + \alpha\frac{\delta-1}{\delta-\alpha}\sup\varphi$

$\Rightarrow \quad D(\varphi) \geq \alpha \frac{\delta-1}{\delta-\alpha} \left(\sup \varphi - E(\varphi) \right) - C_{\alpha,\delta}$.

$\Rightarrow \quad D$ is coercive.

" $\Leftarrow$ " If D is coercive, then $\delta^A > 1$.

pf: For $\forall u \in \mathcal{H}_\omega$, we have
$D(u) \geq \varepsilon(\sup u - E(u)) - C, \quad \varepsilon>0, C>0$

$$D(u) = -\log \int_X e^{-u} dV - E(u)$$
$$= -\log \int_X e^{-u} \frac{dV}{\omega_u^n} \frac{\omega_u^n}{V} - \log V - E(u)$$

Jensen's

$$\leq - \int_X \log\left(e^{-u} \frac{dV}{\omega_u^n}\right) \frac{\omega_u^n}{V} - \log V - E(u)$$

$$= \frac{1}{V}\int_X u\, \omega_u^n + \frac{1}{V}\int_X \log \frac{\omega_u^n}{dV}\, \omega_u^n - \log V - E(u)$$

$$\Rightarrow \frac{1}{V}\int_X \log \frac{\omega_u^n}{dV}\, \omega_u^n \geq E(u) - \frac{1}{V}\int_X u\,\omega_u^n + \varepsilon(\sup u - E(u)) - C$$

entropy of u.

$$\geq \left(1+\frac{\varepsilon}{n}\right)\left(E(u) - \frac{1}{V}\int_X u\,\omega_u^n\right) - C.$$

Claim: $\delta^A([\omega]) \geq 1 + \frac{\varepsilon}{n}$.

Need to show, $\exists\, C>0$ s.t. $\int_X e^{-(1+\frac{\varepsilon}{n})(\varphi - E(\varphi))} dV \leq C$ $\forall \varphi \in \mathcal{H}$.

goal:

i.e. $$-\log\int_X e^{-(1+\frac{\varepsilon}{n})(\varphi-E(\varphi))}dV \geq -C, \forall\varphi.$$

For $\forall$ fixed $\varphi\in\mathcal{H}_\omega$, by Calabi–Yau can find $u\in\mathcal{H}_\omega$ s.t.

$$\omega_u^n = C\cdot e^{-(1+\frac{\varepsilon}{n})(\varphi-E(\varphi))}dV.$$

$$-\log\int_X e^{-(1+\frac{\varepsilon}{n})(\varphi-E(\varphi))}dV = -\log\int_X \frac{e^{-(1+\frac{\varepsilon}{n})(\varphi-E(\varphi))}dV}{\omega_u^n}\frac{\omega_u^n}{V} - \log V$$

$$= -\int_X \log\left(\frac{e^{-(1+\frac{\varepsilon}{n})(\varphi-E(\varphi))}dV}{\omega_u^n}\right)\frac{\omega_u^n}{V} - \log V.$$

$$= (1+\tfrac{\varepsilon}{n})\frac{1}{V}\int_X \varphi\,\omega_u^n + \frac{1}{V}\int_X \log\frac{\omega_u^n}{dV}\,\omega_u^n - \log V - (1+\tfrac{\varepsilon}{n})E(\varphi)$$

$$\geq (1+\tfrac{\varepsilon}{n})\frac{1}{V}\int_X\varphi\,\omega_u^n + (1+\tfrac{\varepsilon}{n})\left(E(u) - \frac{1}{V}\int_X u\,\omega_u^n\right) - (1+\tfrac{\varepsilon}{n})E(\varphi) - \log V - C$$

$$= (1+\tfrac{\varepsilon}{n})\left(\frac{1}{V}\int_X(\varphi-u)\,\omega_u^n + \underline{E(u)-E(\varphi)}\right) - C$$

$$E(u)-E(\varphi) \geq \frac{1}{V}\int_X (u-\varphi)\,\omega_u^n$$

$$\geq 0 - C \geq -C, \quad \forall\varphi\in\mathcal{H}_\omega.$$

$$\Rightarrow \delta^A([\omega]) \geq 1+\frac{\varepsilon}{n} > 1. \qquad \square$$

- $\underline{\delta^A([\omega]) \geq \frac{n+1}{n}\alpha([\omega]).}$

pf: For $\forall\,\alpha\in(0,\alpha[\omega])$, need to show

$$\delta^A([\omega]) \geq \frac{n+1}{n}\alpha([\omega])$$

$$C \geq \log \int_X e^{-\alpha(\varphi - \sup\varphi)} \frac{dV}{\omega_\varphi^n} \frac{\omega_\varphi^n}{V} + \log V$$

$$\geq \int_X \log\left(e^{-\alpha(\varphi-\sup\varphi)} \frac{dV}{\omega_\varphi^n}\right) \frac{\omega_\varphi^n}{V} + \log V$$

$$= \alpha\left(\sup\varphi - \frac{1}{V}\int_X \varphi\,\omega_\varphi^n\right) - \frac{1}{V}\int_X \log\frac{\omega_\varphi^n}{dV}\,\omega_\varphi^n + \log V.$$

$$\Rightarrow \frac{1}{V}\int_X \log\frac{\omega_\varphi^n}{dV}\,\omega_\varphi^n \geq \alpha\left(\sup\varphi - \frac{1}{V}\int_X \varphi\,\omega_\varphi^n\right) - C$$

$$\geq \frac{n+1}{n}\alpha\left(E(\varphi) - \frac{1}{V}\int_X \varphi\,\omega_\varphi^n\right) - C.$$

$$\Rightarrow \delta^A([\omega]) \geq \frac{n+1}{n}\underline{\underline{\alpha}}. \Rightarrow \delta^A([\omega]) \geq \frac{n+1}{n}\alpha([\omega]).$$

$\square$