Lecture 5. The Braid group and the braid recognition algorithm

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Contents

Geometrical definition

Consider the lines $\{y = 0, z = 1\}$ and $\{y = 0, z = 0\}$ in $\mathbb{R}^3$ and choose m points $(1, 0, \epsilon), \ldots, (m, 0, \epsilon)$, $\epsilon = 0, 1$ on each of these lines.

Definition 1.1

An m–strand braid is a set of m non-intersecting smooth paths connecting the chosen points on the first line with the points on the second line (in arbitrary order), such that the projection of each of these paths to Oz represents a diffeomorphism. These smooth paths are called strands of the braid.

An example of a braid is shown in Fig. 1.

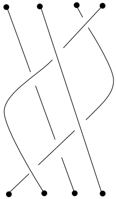

Figure 1: A braid

Definition 1.2

Two braids B_0 and B_1 are equal if they are isotopic; i.e., if there exists a continuous family of braids $B_t, \{t \in \{0,1\}\}$ starting at B_0 and finishing at B_1.

Definition 1.3

The set of all m–strand braids generates a group. The operation in this group is just juxtaposing one braid under the other and rescaling the z–coordinate. The unit element or the unity of this group is the braid represented by all vertical parallel strands. The inverse element for a given braid is just its mirror image; see Fig. 2.

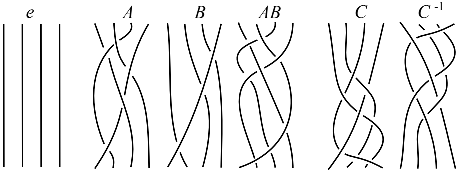

Figure 2: Unity. Operations in the braid group

Definition 1.4

The Artin m–strand braid group is the group of braids with the operation defined above.
Notation: Br(m).

One can consider only braids whose strands connect points with equal x-coordinates.

Definition 1.5

These braids are said to be pure. Pure braids form a subgroup of the braid group. [Art1]
Notation: PB(m).

Remark 1.6

Each braid β generates a permutation σ_β defined by $\sigma_\beta(\mathrm{i}) = \mathrm{j}$ if a strand connects two points $(\mathrm{i}, 0, 0)$ and $(\mathrm{j}, 0, 1)$. Moreover, it gives a group homomorphism ϕ from B(m) to $\mathrm{S_m}$. Note that $\ker(\phi) = \mathrm{PB(m)}$.

Topological definition

Definition 1.7

Given a topological space X, the unordered m–configuration space for X is the space (endowed with the natural topology) of all unordered sets of m pairwise different points of X.
Notation: $B(X, m)$.
Analogously, one can define the m–ordered configuration space.
Notation: $F(X, m)$.

Now, let $X = \mathbb{R}^2 = \mathbb{C}^1$.

Definition 1.8

The m–strand braid group is defined to be isomorphic to the fundamental group $\pi_1(B(X, m))$.

Definition 1.9

The group $\pi_1(F(X, m))$ is called the pure m–strand braid group.

Algebro–geometrical definition

Consider the set of all polynomials of degree m in one complex variable z with leading coefficient equal to one.
Obviously, this set (together with its intrinsic topological structure) is isomorphic to $\mathbb{C}^n$: its coefficients can be considered as its complex coordinates.
Now, delete the space Σ_m of all polynomials that have multiple roots (at least one). We obtain the set $\mathbb{C}^m \backslash \Sigma_m$.

Definition 1.10

The m–strand braid group is the group $\pi_1(\mathbb{C}^m \backslash \Sigma_m)$.

Algebraic definition

Definition 1.11

The m–strand braid group is the group given by the presentation with $(m-1)$ generators $\sigma_1, \ldots, \sigma_{m-1}$ and the following relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

for $|i-j| \geq 2$ and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

for $1 \leq i \leq m-2$.
These relations are called Artin's relations.

Definition 1.12

Words in the alphabet of σ's and σ^{-1}'s will be referred to as braid words.

Theorem 1.13

The four definitions of the braid group Br(m) given above are equivalent.

Proof The easiest part of the proof is to establish the equivalence of the topological and algebro–geometric definitions. Indeed, it is obvious that the two spaces, the space of polynomials of degree m without multiple roots with leading coefficient one and the unordered m–configuration space for $\mathbb{C}^1$, are homeomorphic. Thus, their fundamental groups are isomorphic.

Let us now show the equivalence of the geometrical and topological definitions. As we know, the fundamental group does not depend on the choice of the base point in the connected space. Thus, the base point A of the unordered m–configuration space can be chosen as the set of integer points $(1, 2, \ldots, m)$. Consider the space $\mathbb{R}^3$ as the product $\mathbb{C}^1 \times \mathbb{R}^1$.

With each closed loop, outgoing from A and lying in $B(\mathbb{C}^1, m)$, let us associate a set of lines in $\mathbb{R}^3$ as follows. Each of these (curvilinear) lines represents the motion of a point on the complex line $\mathbb{C}^1$ with respect to the time t, where t is the real coordinate; see Fig. 3.

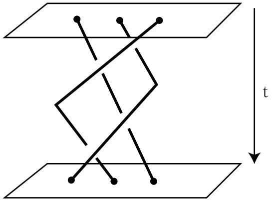

Figure 3: A braid in 3–space

Thus, with each topological braid we have uniquely associated a geometric braid. Obviously, with two homotopic (equal) topological braids we associate the same geometric braids. So, it remains to show the equivalence of geometric and algebraic notions. In order to do this, let us introduce the notion of the planar braid diagram, analogous to the planar link diagram.

To see what this is, let us project a braid on the plane Oxz. In the general case we obtain a diagram that can be described as follows.

Definition 1.14

A braid planar diagram (for the case of m strands) is a graph lying inside the rectangle $[1, m] \times [0, 1]$ endowed with the following structure and having the following properties:

1. Points $(i, 0)$ and $(i, 1), i = 1, \ldots, m$, are vertices of valency one; the other points of type $(x, 0)$ and $(x, 1)$ are not graph vertices.
2. All other graph vertices (crossings) have valency four; opposite edges at such vertices make angles π.
3. Unicursal curves; i.e., lines consisting of edges of the graph, passing from an edge to the opposite one, go from vertices with ordinate one and come to vertices with ordinate zero; they must be descending.
4. Each vertex of valency four is endowed with an over– and undercrossing structure.

Analogously to the planar isotopy of link diagrams, one defines the planar isotopy of braid diagrams.
Obviously, all isotopy classes of geometrical braids can be represented by their planar diagrams. Moreover, after a small perturbation, all crossings of the braid can be set to have different ordinates.
It is easy to see that each element of the geometrical braid group can be decomposed into a product of the following generators σ_i's: the element σ_i for $i = 1, \ldots, m-1$ consists of $m-2$ segments connecting $(k, 1)$ and $(k, 0), k \neq i, k \neq i+1$, and two segments $(i, 0) - (i+1, 1)$, $(i+1, 0) - (i, 1)$, where the latter goes over the first one; see Fig. 4.

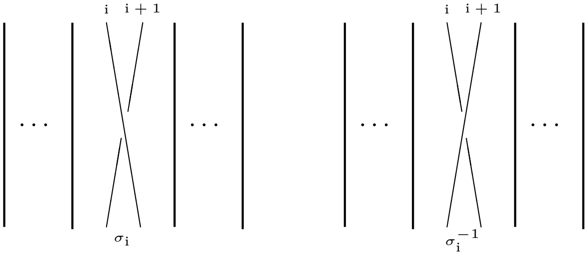

Figure 4: Generators of the braid group

Different braid diagrams can generate the same braid. Thus we obtain some relations in $\sigma_1, \ldots, \sigma_m$.

Let us suppose that we have two equal geometrical braids B_1 and B_2. Let us represent the process of their isotopy in terms of their planar diagrams. Each interval of this isotopy either does not change the disposition of their vertex ordinates, or in this interval at least two crossings have (at some moment) the same ordinate; in the latter case the diagram becomes irregular.

We are interested in those moments where the algebraic description of our braid changes. We see that there are only three possible cases (all others can be reduced to these ones). In the first case (see Fig. 5.a) just one couple of crossings has the same ordinate. In the second case (see Fig. 5.b), two strands are tangent.

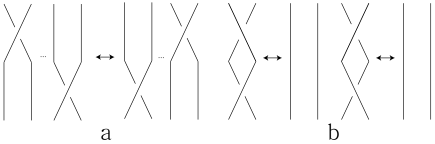

Figure 5: Intuitive expression of braid diagram isotopies 1

In the third case (Fig. 6) we have a triple intersection point.

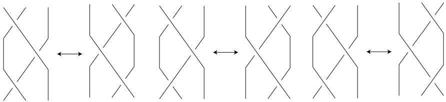

Figure 6: Intuitive expression of braid diagram isotopies 2

It is easy to see that the first case gives us the relation $\sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2$ (this relation is called far commutativity) or an equivalent relation $\sigma_i^{\pm 1}\sigma_j^{\pm 1} = \sigma_j^{\pm 1}\sigma_i^{\pm 1}, |i-j| \geq 2$, in the second case we get $aa^{-1} = 1$ (or $a^{-1}a = 1$), and in the third case we obtain one of the following three relations:

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},$$

$$\sigma_i\sigma_{i+1}\sigma_i^{-1} = \sigma_{i+1}^{-1}\sigma_i\sigma_{i+1}, \quad \sigma_i^{-1}\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}.$$

Obviously, each of the latter two relations can be deduced from the first one. This simple observation is left to the reader as an exercise. This completes the proof of the theorem.

In the m–strand braid group one can naturally define the subgroup PB(m) of pure braids.

The curve algorithm for braids recognition

We shall give a proof of the completeness of one concrete invariant for the braid group elements invented by Artin, see [GM].

We are going to describe the construction of the above mentioned invariant for the braid group Br(n) for arbitrary n.
The invariant to be constructed has a simple algebraic description as a map (non-homeomorphic) from the braid group Br(n) to the n copies of the free group in n generators.

Construction of the invariant

Let us begin with the definition of the notions that we are going to use, and let us introduce the notation.

Definition 2.1

By an admissible system of n curves we mean a family of n non–intersecting non–self–intersecting curves in the upper half plane $\{y > 0\}$ of the plane Oxy such that each curve connects a point having ordinate zero with a point having ordinate one and the abscissas of all curve ends are integers from 1 to n. All points $(i, 1)$, where $i = 1, \ldots, n$, are called upper points, and all points $(i, 0), i = 1, \ldots, n$, are called lower points.

Definition 2.2

Two admissible systems of n curves A and A' are equivalent if there exists a homotopy between A and A' in the class of curves with fixed endpoints lying in the upper half plane such that no interior point of any curve can coincide with any upper or lower point during the homotopy.

Analogously, the equivalence is defined for one curve (possibly, self–intersecting) with fixed upper and lower points: during the homotopy in the upper half plane no interior point of the curve can coincide with an upper or lower point.
In the sequel, admissible systems will be considered up to equivalence.

Remark 2.3

Note that curves may intersect during the homotopy.

Remark 2.4

In the sequel, the number of strands of a braid equals n, unless otherwise specified.

Let β be a braid diagram on the plane, connecting the set of lower points $\{(1,0),\ldots,(n,0)\}$ with the set of upper points $\{(1,1),\ldots,(n,1)\}$. Consider the upper crossing C of the diagram β and push the lower branch along the upper braid to the upper point of it as shown in Fig. 7.

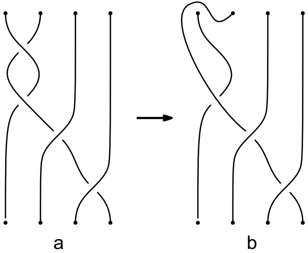

Figure 7: Pushing the upper crossing

Naturally, this move spoils the braid diagram: the result, shown in Figure 7.b is not a braid diagram. The advantage of this "diagram" is that we have a smaller number of crossings.

Now, let us do the same with the next crossing. Namely, let us push the lower branch along the upper branch to the end. If the upper branch is deformed during the first move, we push the lower branch along the deformed branch (see Fig. 8).

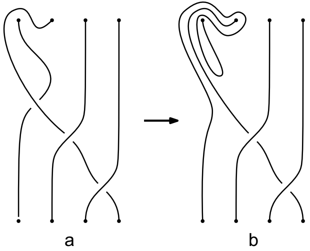

Figure 8: Pushing the next crossing

Reiterating this procedure for all crossings (until the lowest one), we get an admissible system of curves. Denote its equivalence class by f(β).

Theorem 2.5

The function f is a braid invariant; i.e., for two diagrams β, β' of the same braid we have $f(\beta) = f(\beta')$.

Proof. Having two braid diagrams, we can write the corresponding braid–words, and denote them by the same letters β, β'. We must prove that the admissible system of curves is invariant under moves, described in Fig. 5.
The invariance under the commutation relations $\sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2$, is obvious: the order of pushing two "far" branches does not change the result.

The invariants under $\sigma_i\sigma_i^{-1} = e$ can be readily checked; see Fig. 9. In the leftmost part of Fig. 9, the dotted line indicates the arbitrary behaviour for the upper part of the braid diagram. The rightmost part of Fig. 9 corresponds to the system of curves without $\sigma_i\sigma_i^{-1}$.

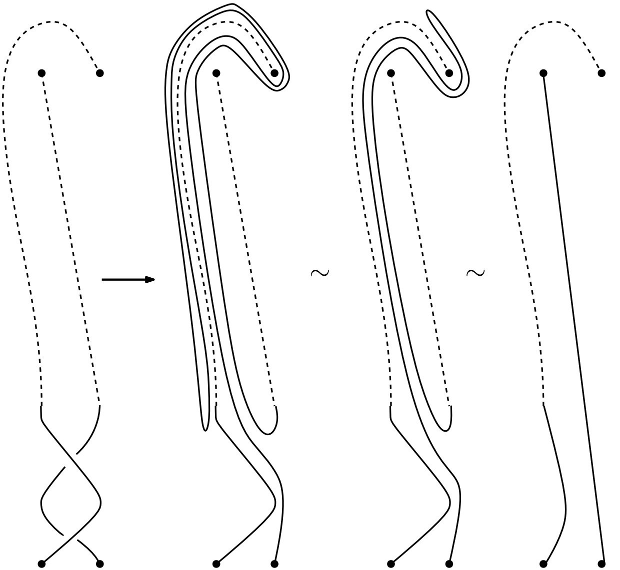

Figure 9: Invariance of f under the move $\sigma_i \sigma_i^{-1} = e$

Finally, the invariance under the transformation $\sigma_i\sigma_{i+1}\sigma_i \to \sigma_{i+1}\sigma_i\sigma_{i+1}$ is shown in Fig. 10. In the upper part (over the horizontal line) we demonstrate the behaviour of $f(A\sigma_i\sigma_{i+1}\sigma_i)$, and in the lower part we show that of $f(A\sigma_{i+1}\sigma_i\sigma_{i+1})$ for an arbitrary braid A. In the middle–upper part, part of the curve is shown by a dotted line. By removing it, we get the upper–right picture which is just the same as the lower–right picture.
The behaviour of the diagram in the upper part A of the braid diagram is arbitrary. For the sake of simplicity it is pictured by three straight lines.
Thus we have proved that $f(A\sigma_i\sigma_{i+1}\sigma_i) = f(A\sigma_{i+1}\sigma_i\sigma_{i+1})$.

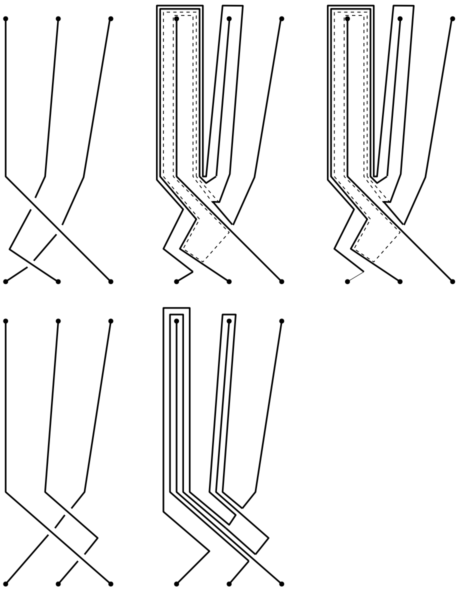

Figure 10: Invariance of f under the move $\sigma_i\sigma_{i+1}\sigma_i) = \sigma_{i+1}\sigma_i\sigma_{i+1}$

Now, let us prove the following lemma.

Lemma 2.6

If for two braids β and β' we have $\mathrm{f}(\beta) = \mathrm{f}(\beta')$ then for each braid γ we obtain $\mathrm{f}(\beta\gamma) = \mathrm{f}(\beta'\gamma)$.

Proof.

The claim $\mathrm{f}(\beta\gamma) = \mathrm{f}(\beta'\gamma)$ follows directly from the construction. Indeed, we just need to attach the braid γ to the admissible system of curves corresponding to β (or β') and then to push the crossings of γ. □

In fact, a much stronger statement holds.

Theorem 2.7 (The main theorem)

The function f is a complete invariant.

To prove this statement, we shall use some auxiliary definitions and lemmas. In order to prove the main theorem, we should be able to restore the braid from its admissible system of curves.
In the sequel, we shall deal with braids whose end points are $(i, 0, 0)$ and $(j, 1, 1)$ with all strands coming upwards with respect to the projection $pr : \mathbb{R}^3 \to \mathbb{R}$ given by $pr(x, y, z) = z$, see Fig. 11.

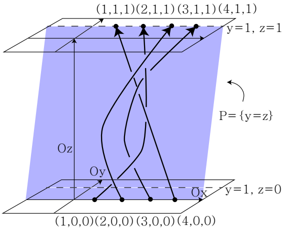

Figure 11: A braid with end points $(i, 0, 0)$ and $(j, 1, 1)$

Consider a braid β and consider the plane $P = \{y = z\}$ in Oxyz. Let us place β in a small neighbourhood of P in such a way that its strands connect points $(i, 0, 0)$ and $(j, 1, 1), i, j = 1, \ldots, n$. Both projections of this braid on Oxy and Oxz are braid diagrams. Denote the braid diagram on Oxy by β.

The next step now is to transform the projection on Oxy without changing the braid isotopy type; we shall just deform the braid in a small neighbourhood of a plane parallel to Oxy.

It turns out that one can change abscissas and ordinates of some intervals of strands of β in such a way that the projection of the transformed braid on Oxy constitutes an admissible system of curves for β.

Indeed, since the braid lies in a small neighbourhood of P, each crossing on Oxy corresponds to a crossing on Oxz. Thus, the procedure of pushing a branch along another branch in the plane parallel to Oxy deletes a crossing on Oxy, preserving that on Oxz. Thus, we have described the geometric meaning of the invariant f.

Definition 2.8

By an admissible parametrisation (in the sequel, all parametrisations are thought to be smooth) of an admissible system of curves we mean a set of parametrisations for all curves by parameters $t_1, \ldots, t_n$ such that at the upper points all t_i are equal to one, and at the lower points t_i are equal to zero.

Any admissible system A of n curves with admissible parametrisation T generates a braid representative: each curve on the plane becomes a braid strand when we consider its parametrisation as the third coordinate. The corresponding braid has end points $(i, 0, 0)$ and $(j, 1, 1)$, where $i, j = 1, \ldots, n$. Denote it by $g(A, T)$.

Lemma 2.9

The result $g(A, T)$ does not depend on T.

Proof.

Indeed, let us consider two admissible parametrisations T_1 and T_2 of the same system A of curves. Let $T_i, i \in [1, 2]$, be a continuous family of admissible parametrisations between T_1 and T_2, say, defined by the formula $T_i = (i - 1)T_1 + (2 - i)T_2$. For each $i \in [1, 2]$, the curves from T_i do not intersect each other, and for each $i \in [1, 2]$ the set of curves $g(A, T_i)$ is a braid, thus $g(A, T_i)$ generates the desired braid isotopy. □

Thus, the function $g(A) \equiv g(A, T)$ is well defined.

Now we are ready to prove the main theorem. First, let us prove the following lemma.

Lemma 2.10

Let A, A' be two equivalent admissible systems of n curves. Then $g(A) = g(A')$.

Proof. Let $A_t, t \in [0,1]$, be a homotopy from A to A'. For each $t \in [0,1]$, A_t is a system of curves (possibly, not admissible). For each curve $\{a_{i,t}, i = 1, \ldots, n, t \in [0,1]\}$ choose points $X_{i,t}$ and $Y_{i,t}$, such that the interval from the upper point (upper interval) of the curve to $X_{i,t}$ and the interval from the lower point (lower interval) do not contain intersection points. Denote the remaining part of the curve (middle interval) between $X_{i,t}$ and $Y_{i,t}$ by $S_{i,t}$. Now, let us parametrise all curves for all t by parameters $\{s_{i,t} \in [0,1], i = 1, \ldots, n\}$ in the following way: for each t, the upper point of each curve has parameter $s = 1$, and the lower point has parameter $s = 0$. Besides, we require that for $i < j$ and for each $x \in S_{i,t}, y \in S_{j,t}$ we have $s_{i,t}(x) < s_{j,t}(y)$.

Proof (continued)

This is possible because we can vary parametrisations of upper and lower intervals on $[0,1]$; for instance, we parametrise the middle interval of the j–th strand by a parameter on $[\frac{j}{n+2}, \frac{j+1}{n+2}]$.
It is obvious that for $t = 0$ and $t = 1$ these parametrisations are admissible for A and A'. For each $t \in [0,1]$ the parametrisation s generates a braid B_t in $\mathbb{R}^3$: we just take the parameter $s_{i,t}$ for the strand $a_{i,t}$ as the third coordinate. The strands do not intersect each other because parameters for different middle intervals cannot be equal to each other.
Thus the system of braids B_t induces a braid isotopy between B_0 and B_1.

So, the function g is well defined on equivalence classes of admissible systems of curves.
Now, to complete the proof of the main theorem, we need only to prove the following lemma.

Lemma 2.11

For any braid β, we have $g(f(\beta)) = \beta$.

Proof.

Indeed, let us place β in a small neighbourhood of the "inclined plane" P in such a way that the ends of β are $(i, 0, 0)$ and $(j, 1, 1)$, $i, j = 1, \dots, n$.
Consider $f(\beta)$ that lies in Oxy. It is an admissible system of curves for β. So, there exists an admissible parametrisation that restores β from $f(\beta)$. By Lemma 2.9, each admissible parametrisation of $f(\beta)$ generates b. So, $g(f(\beta)) = \beta$. □

Algebraic description of the invariant

The general situation in the construction of a complete invariant is the following: one constructs a new object that is in one–to–one correspondence with the described object. However, the new object might also be badly recognisable.

Now, we shall describe our invariant algebraically. It turns out that the final result is very easy to recognise. Namely, the problem is reduced to the recognition problem of elements in a free group. So, there exists an injective map from the braid group to the (n copies of) the free group with n generators that is not homomorphic.

Each braid β generates a permutation. This permutation can be uniquely restored from any admissible system of curves corresponding to β. Indeed, for an admissible system A of curves, the corresponding permutation maps i to j, where j is the ordinate of the strand with the upper point $(i, 1)$. Denote this permutation by $p(A)$. It is obvious that $p(A)$ is invariant under equivalence of A.

Let n be an integer. Consider the free product G of n groups $\mathbb{Z}$ with generators $a_1, \ldots, a_n$. Denote by E_i the right residue classes in G by $\{a_i\}$; i.e., $g_1, g_2 \in G$ represent the same element of E_i if and only if $g_1 = a^k g_2$ for some k.

Definition 2.12

An n–system is a set of elements $e_1 \in E_1, \ldots, e_n \in E_n$.

Definition 2.13

An ordered n–system is an n–system together with a permutation from S_n.

Proposition 2.14

There exists an injective map from equivalence classes of admissible systems of curves to ordered n–systems.

Since the permutation for equivalent admissible systems of curves is the same, we can fix the permutation $s \in S_n$ and consider only equivalence classes of admissible systems of curves with permutation s (i.e., with all lower points fixed depending on the upper points in accordance with s). Thus we only have to show that there exists an injective map from the set of admissible systems of n curves with fixed lower points to n–systems.

To complete the proof of the proposition, it suffices to prove the following.

Lemma 2.15

Equivalence classes of curves with fixed points $(i, 1)$ and $(j, 0)$ are in one–to–one correspondence with E_i.

Proof. Denote $P \setminus \cup_{i=1,\ldots,n} (i, 1)$ by P_n. Obviously, $\pi_1(P_n) \cong G$. Consider a small circle C centered at $(i, 1)$ for some i with the lowest point X on it. Let ρ be a curve with endpoints $(i, 1)$ and $(j, 0)$. Without loss of generality, assume that ρ intersects C in a finite number of points. Let Q be the first such point that one meets while walking along ρ from $(i, 1)$ to $(j, 0)$. Thus we obtain a curve ρ' coming from C to $(j, 0)$. Now, let us construct an element of $\pi_1(P_n, X)$. First it comes from X to Q along C clockwise. Then it goes along ρ until $(j, 0)$. After this, it goes along Ox to the point $(i, 0)$. Then it goes vertically upwards till the intersection with C in X. Denote the constructed element by $W(\rho)$.

Proof (continued)

If we deform ρ outside C, we obtain a continuous deformation of the loop, thus $W(\rho)$ stays the same as the element of the fundamental group. The deformations of ρ inside C might change $W(\rho)$ by multiplying it by a_i on the left side. So, we have constructed a map from equivalence classes of curves with fixed points $(i, 1)$ and $(j, 0)$ to E_i.

The inverse map can be easily constructed as follows. Let W be an element of $\pi_1(P_n, X)$. Consider a loop L representing W. Now consider the curve L' that first goes from $(i, 1)$ to X vertically, then goes along L', after this goes vertically downwards until $(i, 0)$ and finally, horizontally until $(j, 0)$. Obviously, $W(L') = W$. It is easy to see that for different representatives L of W we obtain the same L'. Besides, for $L_1 = a_i L_2$, the curves L'_1 and L'_2 are isotopic. This completes the proof of the lemma.

Thus, for a fixed permutation s, admissible systems of curves can be uniquely encoded by n–systems, which completes the proof of the theorem.
Now, we see that this invariant is quite a simple object: elements of E_i can easily be compared.

Let us describe the algebraic construction of the invariant f in more detail.
Let β be a word–braid, written as a product of generators $\beta = \sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_k}^{\varepsilon_k}$, where each ε_j is either $+1$ or -1; $1 \leq i_j \leq n-1$ and $\sigma_1, \dots, \sigma_{n-1}$ are the standard generators of the braid group $Br(n)$.
We are going to construct the n–system step–by–step while writing the word β. First, let us write n empty words (in the alphabet $a_1, \dots, a_n$). Let the first letter of β be σ_j. Then all words except for the word e_{j+1} should stay the same (i.e., empty), and the word e_{j+1} becomes a_j^{-1}. If the first crossing is negative; i.e., σ_j^{-1} then all words except e_j stay the same and e_j converts to a_{j+1}. While considering each next crossing, we do the following. Let the crossing be $\sigma_j^{\pm 1}$. Let p and q be the numbers of strands coming from the left side and from the right side respectively. If this crossing is positive; i.e., σ_j, then all words except e_q stay the same, and e_q becomes $e_q e_p^{-1} a_p^{-1} e_p$. If it is negative, then all crossings except e_p stay the same, and e_p becomes $e_p e_q^{-1} a_q e_q$. After processing all the crossings, we get the desired n–system.

Example 2.16

For the trivial braid written as $\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}$ the construction operation works as follows:

$$(\mathrm{e},\mathrm{e},\mathrm{e}) \to (\mathrm{e},\mathrm{a}_1^{-1},\mathrm{e}) \to (\mathrm{e},\mathrm{a}_1^{-1},\mathrm{a}_1^{-1}) \to (\mathrm{e},\mathrm{a}_1^{-1},\mathrm{b}_1^{-1}\mathrm{a}_1^{-1}) \to$$
$$\to (\mathrm{e},\mathrm{e},\mathrm{b}_1^{-1}\mathrm{a}_1^{-1}) \to (\mathrm{e},\mathrm{e},\mathrm{b}_1^{-1}) \to (\mathrm{e},\mathrm{e},\mathrm{e}).$$

A priori these words may be non-trivial; they must only represent trivial residue classes, say, $(\mathrm{a}_1,\mathrm{a}_2^2,\mathrm{a}_3^{-1})$.

However, it is not the case.

Proposition 2.17

For the trivial braid, the algebraic algorithm described above gives trivial words.

Proof.

Indeed, the algebraic number of occurencies of a_i in the word e_i equals zero. This can be easily proved by induction on the number of crossings. In the initial position all words are trivial. The induction step is obvious. Thus, the final word e_i equals a_i^p, where $p = 0$. □

From this approach, one can easily obtain the well known action of braids on a free group as follows.

Instead of a set of n words $e_1, \ldots, e_n$, one can consider the words $e_1 a_1 e_1^{-1}, \ldots, e_n a_n e_n^{-1}$.

Since e_i's are defined up to a multiplication by a_i's on the left, the obtained elements are well defined in the free groups. Besides, these elements $\mathfrak{E}_i = e_i a_i e_i^{-1}$ are generators of the free group. This can be checked by a step–by–step confirmation.

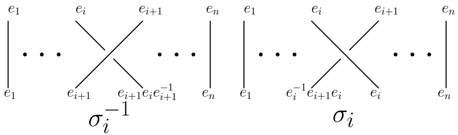

Figure 12: Action corresponding to crossings

Thus, for each braid β we obtain a set $Q(\beta)$ of generators for the braid group. So, the braid β defines a transformation of the free group $\mathbb{Z}^{*n}$.

It is easy to see that for two braids, the transformation corresponding to the product equals the composition of transformation.

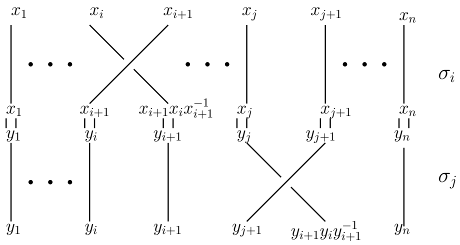

Figure 13: Product of braids and the corresponding action

Thus, one can speak about the action of the braid group on the free group. Since f is a complete invariant, this action has an empty kernel.

Definition 2.18

This action is called the Hurwitz action of the braid group $\mathrm{B_n}$ on the free group $\mathbb{Z}^{*\mathrm{n}}$.

Exercises

1. Show that PB(m) is a normal subgroup in Br(m), and the quotient group Br(m)/PB(m) is isomorphic to the permutation group S(m).
2. Write down the braid words for braid diagrams in Fig. 14.

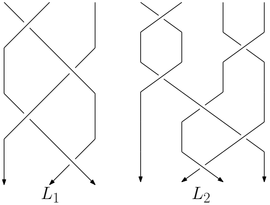

Figure 14:

3. Verify that $\sigma_i\sigma_{i+1}\sigma_i^{-1} = \sigma_{i+1}^{-1}\sigma_i\sigma_{i+1}$ and $\sigma_{i+1}^{-1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i^{-1}$ can be obtained from Artin's relations.

Exercises (continued)

4. Construct an admissible system f(β) for the braid β in Fig. 15.

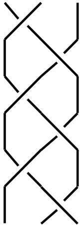

Figure 15:

5. Show that two braid words $\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}$ and $\sigma_1\sigma_3\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1$ are equivalent as braid words and as braid diagrams.
6. (Difficult*) Find a group presentation of the pure braid group PB(m).

Research problem: Δ-groups and Brunnian braids

Let us consider the pure braid group $\mathrm{PB}(X, m) = \pi_1(F(X, m))$ for some topological space, which is considered in page 7. For each $i \in \{1, \ldots, m\}$ there exists a group homomorphism $d_i : \mathrm{PB}(X, m) \to \mathrm{PB}(X, m-1)$, where $d_i(\beta)$ is obtained from $\beta \in \mathrm{PB}(m)$ by deleting i-th strand, for example, see Fig. 16.

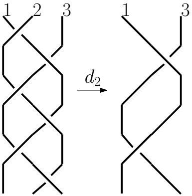

Figure 16: A pure braid on $\mathbb{R}^2$ and the deletion of 2nd strand

It is well known that $F(X)^{\pi_1} = (\mathrm{PB}(X, m), \{d_i\}_{i=1,\ldots,m})$ is a Δ-groups. The elements in $\mathrm{Brunn}(X, m) = \cap_{i=1}^{m} \ker(d_i : \mathrm{PB}(X, m) \to \mathrm{PB}(X, m-1))$ are called Brunnian braids.

In [BCWW] A. J. Berrick, F. R. Cohen, Y. L. Wong and J. Wu it is shown that homotopy groups of S^2 can be calculated by using $F(S^2)^{\pi_1}$ and $\{Brunn(S^2, m)\}_{m=1}^{\infty}$, in other words, the Brunnian braids can show topological properties of the topological space, where the Brunnian braids are defined. One can find that the above statement can be proved by using a chain complex, so called Moore complex, and it can be obtained not only from $F(S^2)^{\pi_1}$ and $\{Brunn(S^2, m)\}_{m=1}^{\infty}$, but also $F(X)^{\pi_1}$ and $\{Brunn(S^2, m)\}_{m=1}^{\infty}$ for any topological space X. Then the following question is very natural:

Question How can we study topological spaces X by using $F(X)^{\pi_1}$ and $Brunn(X, m)$?

On the other hand, to study Brunnian braids is also interesting problem. In [3] Brunnian briads are studied by using groups G_n^k, especially, G_n^2 and G_n^3, which correspond to a configuration spaces with good codimension 1 property. In the future they are covered in our lecture.
Question How can we study topological spaces X by using groups G_n^k?

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