The Poisson σ-model and semiclassical asymtotics for integrable systems.

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Chern: a great geometer of the 20th century

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- Classical Poisson σ-model as a topological gauge field theory
- Quantization and gauge fixing
- Applications:
  - representation groupoid for Kontsevich's $\ast$-product
  - universal semiclassical formula for eigenfunctions for quantum integrable systems

The Poisson σ-model, classical topological field theory

Fields:

mappings $(X,\eta): T\Sigma \longrightarrow T^*M$

$$X: u \in \Sigma \mapsto X(u) \in M$$

$$\eta: t \in T_u\Sigma \mapsto (\eta(u), t) \in T^*_{X(u)}M$$

In local coordinates (on M),

$$X(u) = (X^1(u), \dots, X^n(u))$$

$$\eta(u) = (\eta_1(u), \dots, \eta_n(u)), \quad \eta_i(u) \in \Omega^1(\Sigma)$$

- Σ - space time, a smooth 2d manifold, possibly with boundary
- M - target space, with a Poisson structure $p \in \Gamma(\Lambda^2 TM)$, s.t.

$$\{f, g\} = p(df \wedge dg)$$

is a Lie algebra structure on $C^\infty(M)$ (Jacobi identity).

- The action functional (Schaller, Strobl, 1994)

$$S[X,\eta] = \int_\Sigma \Big(\sum_i \eta_i \wedge dX^i + \frac{1}{2} \sum_{ij} p^{ij}(X)\, \eta_i \wedge \eta_j \Big)$$

The first variation:

$$\delta S[X,\eta] = \int_\Sigma \delta\eta_i \wedge \big(dX^i + p^{ij}(X)\,\eta_j\big) +$$

$$+ \frac{1}{2} \int_\Sigma \partial_k p^{ij}(X)\, \eta_i \wedge \eta_j\, \delta X^k + \int_\Sigma d\eta_i\, \delta X^i -$$

$$- \int_{\partial\Sigma} \eta_i\, \delta X^i$$

Euler-Lagrange equations:

- $dX^i + p^{ij}(X)\,\eta_j = 0$
- $d\eta_k + \frac{1}{2} \partial_k p^{ij}(X)\, \eta_i \wedge \eta_j = 0$

$$\delta_{\text{boundary}} S'(X,\eta) = -\int_{\partial\Sigma} \eta_i\, \delta X^i ,$$

Defines 1-form on $F_{\partial\Sigma} = \{(X,\eta) : \partial\Sigma \to T^*M\}$

$$\alpha_{\partial\Sigma} = -\sum_i \int_{\partial\Sigma} \eta_i\, D X^i$$

$$\delta_{\text{Boundary}} S' = \langle (\delta X, \delta\eta), \alpha_{\partial\Sigma} \rangle$$

$\omega_{\partial\Sigma} = D\alpha_{\partial\Sigma}$ – natural symplectic structure on $F_{\partial\Sigma}$

Boundary conditions

Variational boundary conditions:

- $(X,\eta)|_{\partial\Sigma} \in L$ s.t.

$$\delta_{\text{boundary}} S' \big|_L = 0$$

- i.e. $\alpha_{\partial\Sigma}\big|_L = 0$

and we want L to be maximal with this property.

L = a Lagrangian α- exact submanifold in $F_{\partial\Sigma}$

Examples:

1) $\eta|_{\partial\Sigma} = 0$, Cattaneo-Felder b.c.

$\Rightarrow$ Kontsevich's $*$- product after quantization.

2) $\mathcal{L} \subset M$, $N^*\mathcal{L}$ - conormal bundle to $\mathcal{L}$

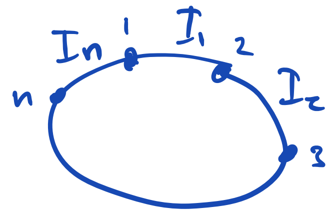
$N^*\mathcal{L} \subset T^*M$ Lagrangian.

corresponding boundary condition (on $I \subset \mathcal{L}$)

$$X(u) \in \mathcal{L} \ , \quad \eta(u) \in N^*_{X(u)}\mathcal{L} \times T^*_u I$$

Lagrangian boundary conditions

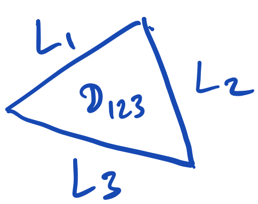
For each connected component of $\partial\Sigma$ and a stratification $\partial\Sigma = I_1 \cup \dots \cup I_n$



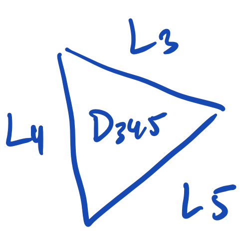
Stratified boundary conditions: $(x,y)\big|_{I_i} \in L_i \subset F_{\partial M}$ Lagrangian

Gluing

Example:

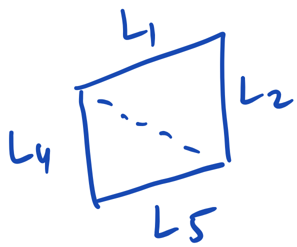


$\rightsquigarrow S_{D_{123}}[x,y]$, its critical points and critical values for fixed L_1, L_2, L_3



$\rightsquigarrow S_{D_{345}}[x,y]$ —"—

Can we construct critical points and critical values of $S_{D_{1245}}[x,y]$ from



critical points and critical values of $S_{D_{123}}$ and $S_{D_{345}}$?

Yes if L_3 comes in a family of Lagrangians i.e. when L_3 is a generic fiber of a Lagrangian fibration

$$\begin{array}{ccc} T^*M & \hookleftarrow & L_3 \\ \downarrow & & \\ B_3 & & \end{array}$$

$$c_{1245} = \left\{ \begin{array}{l} \text{extrema of } \left(S_{123}(c_{123}) + S_{345}(c_{345}) \right) \text{ when} \\ \text{we vary } L_3 \end{array} \right\}$$

$$S_{1245}(c_{1245}) = \left(S_{123}(c_{123}) + S_{345}(c_{345}) \right) \Big|_{L_3^{crit}}$$

This is a classical (with corners) version of Atiyah gluing principle in TQFT.

Gauge symmetry

In local coordinates, gauge symmetry vector fields are:

$$\delta_\beta X^i = \sum_j p^{ij}(X)\beta_j ,$$

$$\delta_\beta \eta_i = -d\beta_i - \partial_i p^{jk}(X)\eta_j \beta_k$$

β_i - components of $\beta \in \Omega^1(M)$,

Thm (CF)

$$\delta_\beta S = -\sum_i \int_{\partial\Sigma} \beta_i dX^i$$

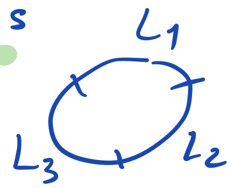
Note: • vector fields $\{\delta_\beta\}$ do not form a Lie algebra on all fields

• Gauge transformations form a Lie algebra on solutions to EL equations

Quantization

The idea:

- impose boundary conditions

$(X,\eta)|_{\partial\Sigma} \in L$ ()

- consider the "path integral"

$$Z_\Sigma(L) = \int_{(X,\eta)|_{\partial\Sigma}\in L} e^{\frac{i}{\hbar}\substack{S[X,\eta]\\ G[X,\eta]}} \, \mathcal{D}X \, \mathcal{D}\eta$$

If $S[X,\eta]$ with b.c. would be nondegenerate we can define it as:

$$Z_\Sigma(L)^{\text{formal}} = \frac{C}{\sqrt{\mathrm{Hess}(S_c^{(2)})}} e^{i\frac{S(c)}{\hbar}} \left(1 + \sum \text{Feyn. diagr.}\right)$$

By analogy with

$$\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}S(x)} g(x)\,dx = C \frac{e^{\frac{i}{\hbar}S(c) + \frac{i\pi}{4}\operatorname{sign}(S^{(2)}(c))}}{\sqrt{|\det(S^{(2)}(c))|}} \times$$

$$\times \left(g(c) + \sum \text{Feynman diagrams} \right)$$

assuming finite support of g, and $f(x)$ has unique critical point c.

But $S[X,\eta]$ is degenerate (Gauge symmetry).

The idea of defining perturbative path integral: rewrite the integral not as an integral over gauge classes (too singular), but over a large space $\mathcal{F}$ (Fadder-Popov; Batalin-Vilkovisky).

FP is for gauge symmetries with respect

to group actions; BV is more general.

$$\int_{F_\Sigma} e^{i\frac{S[X,\eta]}{\hbar}} DX\,D\eta = \int_{L_{BV} \subset \mathcal{F}_\Sigma} e^{\frac{i}{\hbar} S_{BV}(\tilde{X},\tilde{\eta})} \mathcal{D}\ell$$

Lagrangian submanifold (L_{BV})

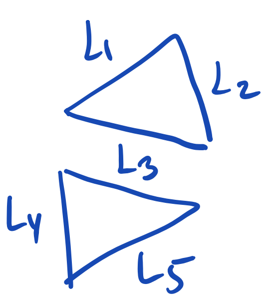
supermanifold "extending F_Σ" odd symplectic ($\mathcal{F}_\Sigma$)

when S_{BV} satisfies the master equation

$$\{S_{BV}, S_{BV}\} = 0 \;,\; \left(Q_{BV} = \{S_{BV}, \cdot\},\; Q_{BV}^2 = 0\right)$$

the integral does not depend on deformations of L_{BV} (A. Schwarz); $S_{gf} = S_{BV}\big|_{L_{BV}}$ is nondegenerate.

Gluing:



$$\int^{\text{formal}}_{\{L_3\}} Z^{\text{formal}}_{D_{123}}(L_1, L_2, L_3)\, Z^{\text{formal}}_{D_{345}}(L_3, L_4, L_5)\, DL_3 =$$

$$= Z^{\text{formal}}_{D_{1245}}(L_1, L_2, L_4, L_5), \ \dots$$

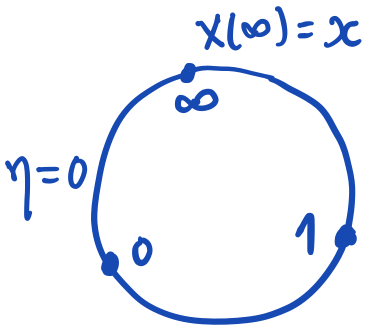
Functorial topological quantum field theory. Work in progress (Cattaneo, Mnev, ..., R, ...).

Applications

1) Kontsevich $*$-product on $C^\infty(M)[[h]]$:

(Kontsevich, 1997) (Cattaneo, Felder, 2001)
$*$-product BV-interpretation

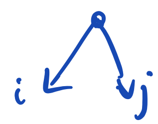
$$(f * g)(x) = \int_{X(\infty)=x} e^{\frac{i}{\hbar} S[X,\eta]} f(X(0))\, g(X(1))\, \mathcal{D}X\, \mathcal{D}\eta$$



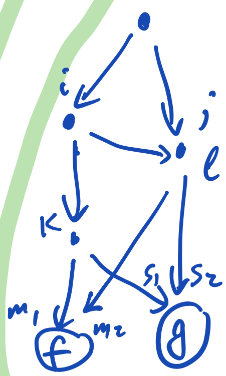
perturbative (semiclassical) expansion near critical point $\begin{cases} X(u) = x \\ \eta(u) = 0 \end{cases}$

$$= f(x)g(x) + \frac{i\hbar}{2}\{f,g\} + \sum_{n\geq 2} \frac{\hbar^{n}}{n!\,2^n}\left(\frac{1}{2\pi}\right)^{2n}$$

$$\cdot \int \wedge_{j=1}^{n} d\phi(u_j, u_{v_1(j)}) \wedge d\phi(u_j, u_{v_2(j)}) \cdot D_\Gamma(f\otimes g)$$

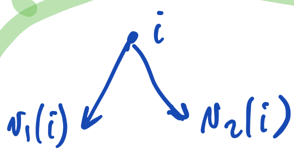


$\rightsquigarrow p^{ij}(x)\,\partial_i\wedge\partial_j$



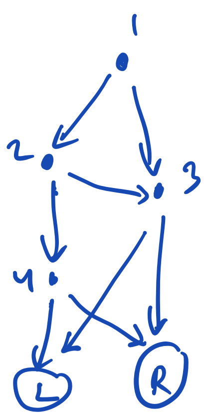
$$\rightsquigarrow p^{ij}(x)\,\partial_i p^{kl}(x)\;\partial_j\partial_l p^{m_2 s_2}$$

$$\partial_k p^{m_1 s_1}\;\partial_{m_1}\partial_{m_2} f\;\;\partial_{s_1}\partial_{s_2} g$$



choose enumeration of vertices

$u_i \in H$, $\quad u_L = 0,\ u_R = 1,\quad u_{L,R}\in\partial H$



$$d\phi(u_1,u_2)\wedge d\phi(u_1,u_3)\wedge$$

$$\wedge\ d\phi(u_2,u_4)\wedge d\phi(u_2,u_3)\wedge$$

$$\wedge\ d\phi(u_3,u_L)\wedge d\phi(u_3,u_R)\wedge$$

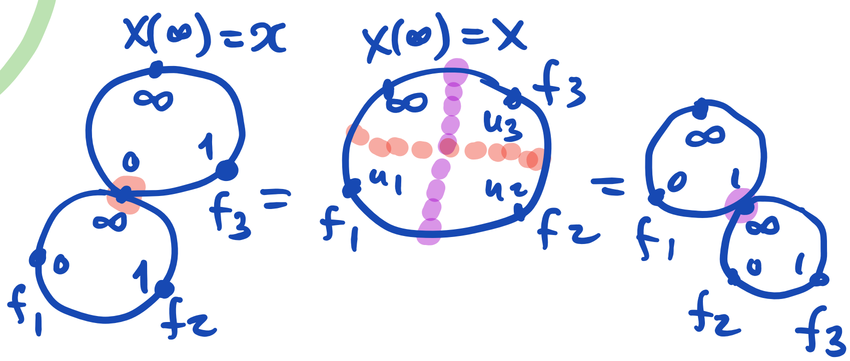
$$\wedge\ d\phi(u_4,u_L)\wedge d\phi(u_4,u_R)$$

Upper half-plane models the disc,

$$\phi(z,w) = \frac{1}{2i} \ln \left(\frac{(z-w)(z-\bar{w})}{(\bar{z}-\bar{w})(\bar{z}-w)} \right)$$

<u>This $*$-product is associative:</u>

$$f_1 * (f_2 * f_3) = \\ = (f_1 * f_2) * f_3$$



(a version of gluing)

2) <u>Representation groupoid for Kontsevich's $*$-product</u>

A – associative algebra $(C^\infty(M)[[h]], *\text{-product})$

- $(V, \pi_V : A \to \mathrm{End}(V))$ is a representation of A in V if $\pi_V(ab) = \pi_V(a)\,\pi_V(b)$

- <u>Representation groupoid</u>:

Vector spaces $\{V_i\}$ $\quad V_i \simeq V_j$ and

$\pi_{V_iV_j} : A \to \mathrm{Hom}_k(V_j, V_i)$

s.t.

$$\pi_{V_iV_j}(a\,b) = \pi_{V_iV_k}(a)\,\pi_{V_kV_j}(b)$$

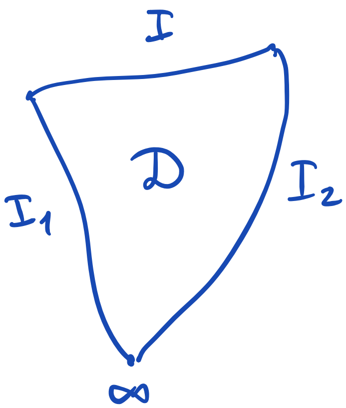
for "admissible" triples (i,k,j).

- Assume we have two transversal Lagrangian fibrations on (M,ω):

$$\begin{tikzcd} & M_{2n} \arrow[dl] \arrow[dr] & \\ B_1 & & B_2 \end{tikzcd}$$

$\leftarrow$ generic fibers are Lagrangian and transversal

$\mathcal{L}^{(1)}_{b_1} \cap \mathcal{L}^{(2)}_{b_2} =$ finitely many points



- $\eta|_{I} = 0$,

- $X|_{I_1} \in \mathcal{L}_1$, $\eta(u) \in N^*_{X(u)}\mathcal{L}_1 \otimes T^*_u I_1$,
  $u \in I_1$, $\mathcal{L}_1 \equiv \mathcal{L}^{(1)}_{e_1}$,

- $X|_{I_2} \in \mathcal{L}_2$, $\eta(u) \in N^*_{X(u)}\mathcal{L}_2 \otimes T^*_u I_2$,
  $u \in I_2$, $\mathcal{L}_2 \equiv \mathcal{L}^{(2)}_{e_2}$.

- $X(\infty) = a \in \mathcal{L}_1 \cap \mathcal{L}_2$

The gauge fixed action $S_{gf} = S_{BV}|_{L_{BV}}$

critical point

BV fields

$X(u) = a$, $\eta(u) = 0$, ...

Consider (Cattaneo, Mnev, R, 2018)

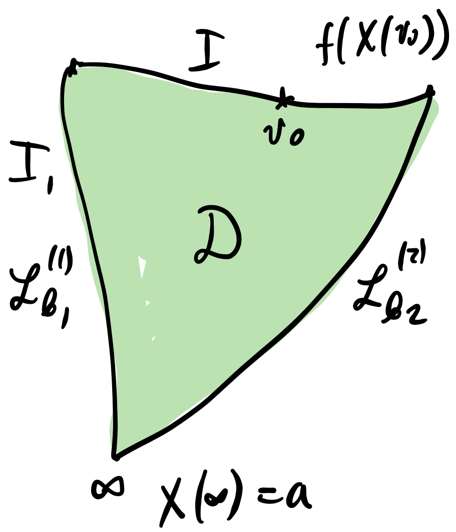
$$\left(\psi^{(2)}_{B_2}, \hat{f}\, \psi^{(1)}_{B_1}\right) = \sum_a \int e^{\frac{i}{\hbar} S^{BV}_{gf}} f(X(v_0))\, \mathcal{D}\xi\, \mathcal{D}\eta\, \mathcal{D}\beta\, \mathcal{D}\gamma$$

formal
(a)

"fluctuations"

$X(u) = a + \xi(u)$

BV-fields



$f \in C^\infty(M)$

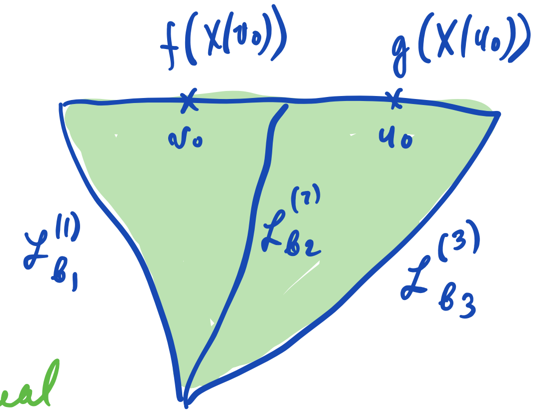
$$\left(\psi^{(2)}_{B_2}, \hat{f}\, \psi^{(1)}_{B_1}\right) = \frac{C}{(2\pi\hbar)^{n/2}} \sum_{a \in \mathcal{L}^{(1)}_{B_1} \cap \mathcal{L}^{(2)}_{B_2}} \left| \det\left(\frac{\partial^2 S_{\gamma_1\gamma_2}(a)}{\partial b_1^i\, \partial b_2^j} \right) \right|^{\frac{1}{2}} \sqrt{|db_1|\,|db_2|}$$

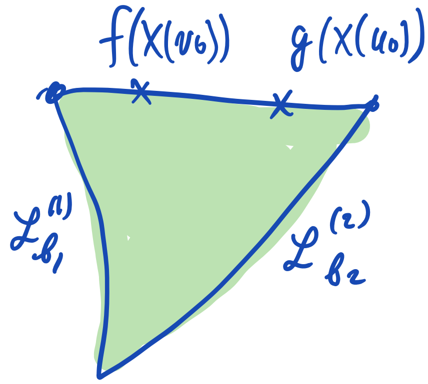
$$\exp\left(\frac{i}{\hbar} S_{\gamma_1\gamma_2}(a) + i\frac{\pi}{2} \mu_{\gamma_1\gamma_2}(a) \right) \left(f(a) + \sum_\Gamma \hbar^{d(\Gamma)} w_\Gamma \cdot D_\Gamma(f,a) \right)$$

These are semiclassical "matrix elements" of

$\pi_{B_1 B_2}(\hat{f}) : \mathcal{H}_{B_2} \to \mathcal{H}_{B_1}$ (space of geom quant. for real polariz.)

Representation groupoid property:

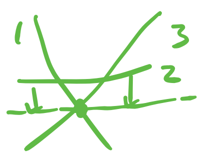
$\int$ 

= 

formal over b_2

in the vicinity of

$b_2(b_1,b_3)$ = triple intersection

$\mathcal{L}^{(1)}_{b_1}\cap\mathcal{L}^{(2)}_{b_2}\cap\mathcal{L}^{(3)}_{b_3}$ 

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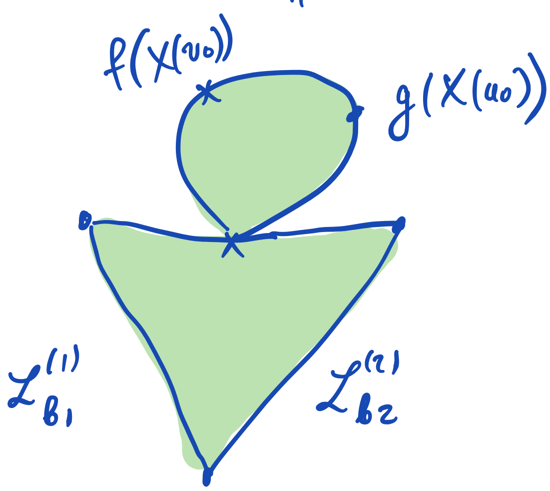
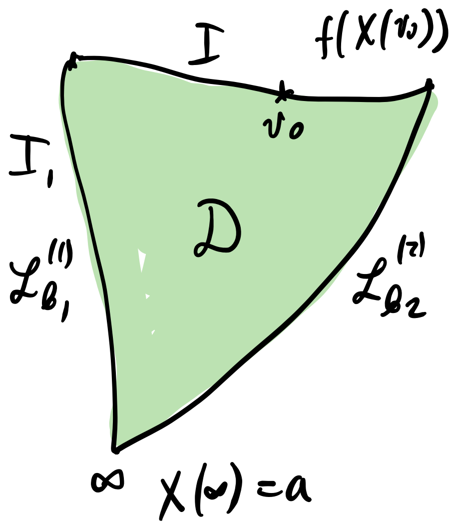


Or:

$$\int_{\text{formal } b_2} \underbrace{(\psi^{(2)}_{b_2}, \hat{f}\,\psi^{(1)}_{b_1})(\psi^{(3)}_{b_3}, \hat{g}\,\psi^{(2)}_{b_2})}_{\substack{\text{density in } b_2 \\ \frac{1}{2}\text{-density in } b_1, b_3}} = (\psi^{(3)}_{b_3}, \widehat{f*g}\,\psi^{(1)}_{b_1})$$

i.e. representation groupoid for Konts. $*$-product.

Integrable systems



$$\begin{tikzcd}
\mathcal{L}^{(1)}_{B_1} \arrow[r] & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \mathcal{L}^{(2)}_{B_2} \arrow[l] \\
B_1 & & B_2
\end{tikzcd}$$

$\mathcal{L}^{(1)}_{B_1}$ transversal to $\mathcal{L}^{(2)}_{B_2}$

Two real polarizations of M, two classical integrable systems.

1) If $f = \pi^*(g_1)$, $g_1 \in C^\infty(B_1)$

we can move v_0 to the left corner and

$$(\psi^{(1)}_{B_1}, \hat{f}\, \psi^{(2)}_{B_2}) = g_1(B_1)(\psi^{(1)}_{B_1}, \psi^{(2)}_{B_2})$$

2) If $f = \pi^*(g_2)$, $g_2 \in C^\infty(B_2)$

$$(\psi^{(1)}_{B_1}, \hat{f}\, \psi^{(2)}_{B_2}) = (\psi^{(1)}_{B_1}, \psi^{(2)}_{B_2})\, g_2(B_2)$$

i.e. $\Psi^{(1)}_{b_1}$, $\Psi^{(2)}_{b_2}$ are eigenfunctions of two integrable systems.

Poisson σ-model $\rightarrow$ universal semiclassical quantization of integrable systems (eigenfunctions in all orders in $\hbar$)