

Lecture 1: Reidemeister moves, colouring invariant, linking number

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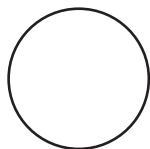
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Definition 1.1

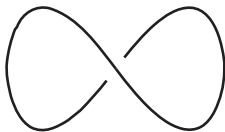
By a knot we mean a smooth embedding of the circle S^1 in \mathbb{R}^3 (or in the sphere S^3) as well as the image of this embedding.

Definition 1.2

Two knots K_1 and K_2 are equivalent, if there exists an orientation preserving diffeomorphism of \mathbb{R}^3 (or S^3) to itself which maps K_1 to K_2 .



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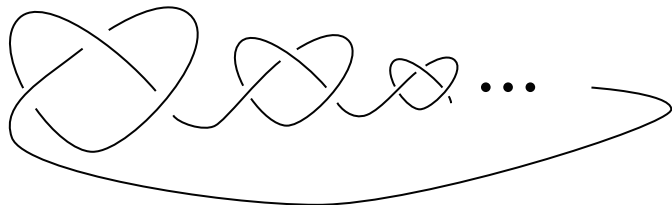


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. 1: Examples of knots

Remark 1.3

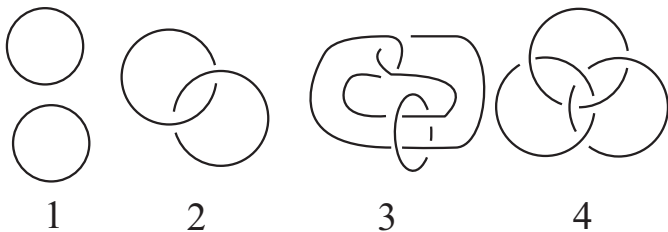
In Knot theory we consider “smooth” embeddings of circles. If we consider just embeddings of circles in \mathbb{R}^3 , then we meet with interesting knots, which is called wild knots, depicted in Fig. 2.



. 2: A wild knot

Definition 1.4

A link is a finite ordered collection of knots $K_1 \sqcup \dots \sqcup K_n$ which do not intersect each other, equivalently, a link is a smooth embedding of $S^1 \sqcup \dots \sqcup S^1$ in \mathbb{R}^3 . Each knot K_i is called a component of the link.



. 3: Example of links

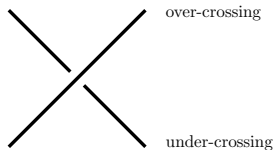
Definition 1.5

Two links $L = K_1 \sqcup \cdots \sqcup K_m$ and $L' = K'_1 \sqcup \cdots \sqcup K'_m$ are equivalent if the following two conditions hold:

- 1 $m = n$
- 2 There exists an orientation-preserving diffeomorphism of \mathbb{R}^3 that maps K_i to K'_i for each $i \in \{1, \dots, n\}$.

Definition 1.6

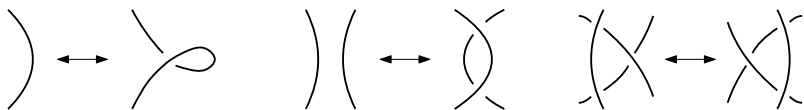
For a link $L \in \mathbb{R}^3$ consider a projection of L on a plane along a vector v in general position. The image of the projection is called a shadow. For each double point of the shadow we can give over-under information with respect to v as described in Fig. 4. We call a shadow with over-under structure a knot diagram.



. 4: Classical crossing of a diagram

Theorem 1.7

Two oriented diagrams D_1 and D_2 of smooth links generate equivalent links if and only if D_1 can be transformed into D_2 by using a finite sequence of planar isotopy and the three Reidemeister moves $\Omega_1, \Omega_2, \Omega_3$, shown in Fig. 5.

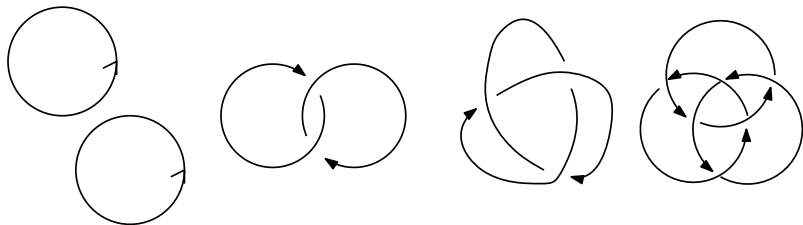


. 5: Reidemeister moves $\Omega_1, \Omega_2, \Omega_3$

Oriented link

Definition 1.8

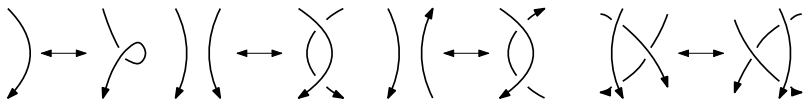
An oriented knot (or link) is a smooth embedding of an oriented circle (or disjoint union of oriented circles) in \mathbb{R}^3 . Roughly speaking, each component of a link is oriented.



. 6: Oriented links

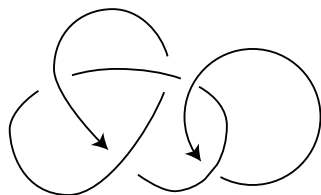
Theorem 1.9

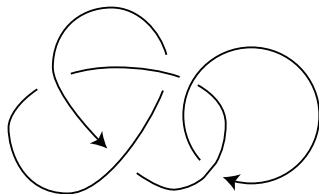
Two oriented diagrams D_1 and D_2 of smooth links generate equivalent links if and only if D_1 can be transformed into D_2 by using a finite sequence of planar isotopy and the three Reidemeister moves $\Omega_1, \Omega_2, \Omega_3$, shown in Fig. 5.



. 7: Oriented Reidemeister moves $\Omega_1, \Omega_2, \Omega_2', \Omega_3$

Orientation reversal



$$L_1 \sqcup L_2$$


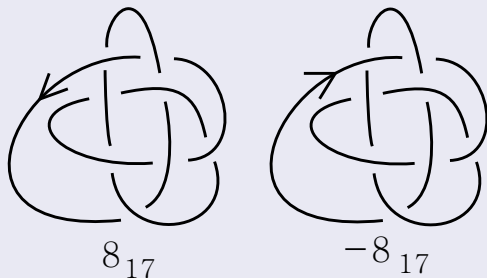
$$L_1 \sqcup -L_2$$

. 8: Orientation reverse

Definition 1.10

Let K be an oriented knot. If K is equivalent to $-K$, then we call it invertible, otherwise non-invertible

Example 1.11



. 9: 8_{17} knot – non-invertible knot

For a knot K many knot invariants cannot recognize the orientation reversal. The existence of (infinitely many) non-invertible knots is proved by H.F. Trotter by using “pretzel knots” and “the knot group”.

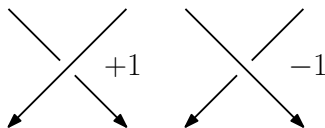
Definition 1.12

Let $L = L_1 \sqcup \cdots \sqcup L_n$ be an oriented link of n components.

There are two types of classical crossings and we assign $+1$ or -1 for each classical crossing as described in Fig. 10.

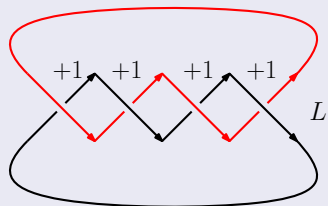
We call it the sign of the crossing c and denote by $\text{sign}(c)$.

For L_i and $L_j, i \neq j$, the linking number $\text{lk}(L_i, L_j)$ between i and j -th components is defined by $\text{lk}(L_i, L_j) = \frac{1}{2} \sum_{c \in L_i \cap L_j} \text{sign}(c)$.



. 10: Positive and negative crossings

Example 1.13



$$\cdot 11: \text{lk}(L_1, L_2) = \frac{1}{2} \sum \text{sign}(c) = 2$$

Remark 1.14

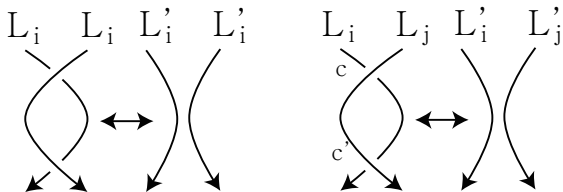
The linking number is found by C.F. Gauss, which is calculated by an explicit formula as a double line integral, the Gauss linking integral:

$$\begin{aligned} \text{lk}(L_1, L_2) &= \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \cdot (d\mathbf{r}_1 \times d\mathbf{r}_2) \\ &= \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{\det \dot{\gamma}_1(s), \dot{\gamma}_2(s), \gamma_1(s) - \gamma_2(t)}{|\gamma_1(s) - \gamma_2(t)|^3} ds dt, \end{aligned}$$

where $\gamma_1, \gamma_2 : S^1 \rightarrow \mathbb{R}^3$. This formula is generalized by M.L. Kontsevich, and it is called the Kontsevich integral.

Theorem 1.15

Let $L = L_1 \sqcup \cdots \sqcup L_n$ be an oriented link of n components. For each pair $i, j \in \{1, \dots, n\}$ the linking number $\text{lk}(L_i, L_j)$ is invariant under Reidemeister moves.



$$\begin{aligned} \text{lk}(L_i, L_j) &= \text{lk}(L'_i, L'_j) & \text{lk}(L_i, L_j) &= \text{lk}(L'_i, L'_j) + \text{sign}(c) + \text{sign}(c') \\ & & &= \text{lk}(L'_i, L'_j) \end{aligned}$$

. 12: $\text{lk}(L_1, L_2) = \text{lk}(L'_1, L'_2)$ under Reidemeister move 2

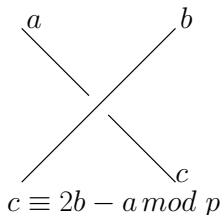
Definition 1.16

Let D be a diagram and $A(D)$ the set of all arcs. For a prime number p the Fox p -colouring (or simply p colouring) is a function $f : A(D) \rightarrow \mathbb{Z}_p$ satisfying the condition described in Fig. 13.

If $f(a) = r$ for every $a \in A(D)$ for some $r \in \mathbb{Z}_p$, then it is called a trivial colouring.

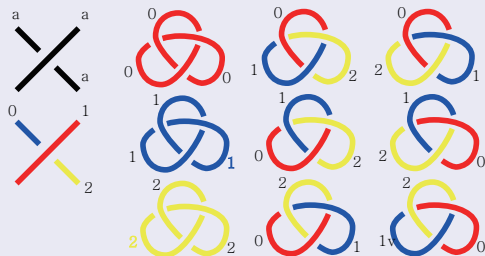
If D has non-trivial p colouring, then we call it p colourable.

The number of Fox p -colourings on D is denoted by $c_p(D)$.



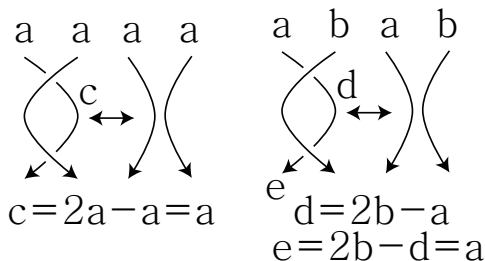
. 13: Colouring condition

Example 1.17

. 14: Proper colorings of trefoil by \mathbb{Z}_3

Theorem 1.18

Let D be a link diagram. For each prime number p $c_p(D)$ is an invariant under Reidemeister moves.



. 15: $c_p(D)$ is an invariant under the second Reidemeister move

Some digressions 1

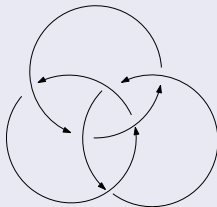
Remark 1.19

We can think about knots in \mathbb{R}^4 , i.e. embedded S^1 in \mathbb{R}^4 . But, it is well-known that every knots in \mathbb{R}^4 is isotopic to the trivial knot in \mathbb{R}^4 , i.e. a boundary of a 2-dimensional disc in $\mathbb{R}^3 \subset \mathbb{R}^4$ (exercise). But embedded “surfaces” in \mathbb{R}^4 give a non-trivial theory, so called surface knot theory.

Some digressions 2

Remark 1.20

The link, shown in Fig. 16 is called Borromean rings. It has the following property: When we remove one of components, then we obtain a trivial link diagram, but it is non-trivial link. Links satisfying the above property are called Brunnian links. To show non-triviality of Brunnian links is not easy task. This problem will be asserted in the future with link homotopies.

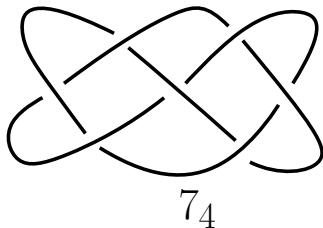


. 16: Borromean rings

Exercises

- 1 Find an unknot diagram D such that no decreasing Ω_1 and Ω_2 and no Ω_3 can be applied to D .
- 2 Verify that the left-handed and right-handed figure-eight knots are isotopic.
- 3 Show that every “knot” diagram with 0, 1 or 2 crossings is equivalent to a knot diagram with no crossing.








- 4 Show that a descending knot diagram is equivalent to a knot diagram with no crossing. Show that every link diagram can be changed to a knot diagram of trivial knots by crossing change.
- 5 Define an unknotting number of a knot by the minimal number of crossing change for a given knot to be trivial knot.
 - Verity that the unknotting numbers of the trefoil knot and the figure-eight knot are at most 1.
 - Verity that the unknotting numbers of the knot 7_4 described in Fig. 17 is at most 2. Is it true that the unknotting number of 7_4 is 2?



. 17: A diagram of 7_4 knot

- 6 Verify that for a link L the linking number $\text{lk}(L_i, L_j)$ is an invariant under Reidemeister move 3.
- 7 Let $L = L_1 \sqcup L_2$ be a link. Show that $\text{lk}(L_1, L_2) = \text{lk}(L_2, L_1)$.
- 8 Let $L = L_1 \sqcup L_2$ be a link. Show that $-\text{lk}(L_1, L_2) = \text{lk}(-L_1, L_2)$.
- 9 Calculate linking numbers for the following links
 - Hopf link
 - Borreomeo link
 - Whitehead link
- 10 Calculate the number of 3-colourings of the following knots:
 - Right and left handed trefoil knots
 - Figure-eight knot
 - Borromean link

- 1.1 Show that the number of proper colourings by p colours is equal to p^n for some $n \in \mathbb{N}$. (Hint: Use linear algebra.)
- 1.2 Prove that the unknotting number is greater than or equal to $\ln_3(c_3(K)) - 1$.
- 1.3 (Problem)
 - Estimate the number of Reidemeister moves needed to transform two isotopic diagrams having n crossings and m crossings.
 - Estimate the number of crossings of diagrams, which appeared when we transform a diagram having n crossings to an equivalent digram and m crossings.

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