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1 Convergence of percolation interface

Let $(D; x, c)$ be a Dobrushin domain, i.e. $D \subset \mathbb{C}$ is a non-empty bounded simply connected domain with two boundary points x, c such that ∂D is a simple closed curve. Denote by $\overline{xc}$ the boundary arc from x to c in counterclockwise order. We will consider the convergence of percolation interface curves in this note. The metric on the curves is the following:

$$\mathrm{d}(\gamma_1, \gamma_2) := \inf \sup_{t \in [0,1]} |\gamma_1(t) - \gamma_2(t)|_{\mathbb{R}^2}, \tag{1.1}$$

where the infimum is taken over all parametrizations $\gamma_1, \gamma_2 : [0, 1] \to \mathbb{C}$. Suppose $\{(D^{\delta_n}; x^{\delta_n}, c^{\delta_n})\}_{n=1}^{\infty}$ is a sequence of discrete Dobrushin domains on hexagonal lattice with mesh-size δ_n. Suppose it converges to $(D; x, c)$ in the following sense:

- $x^{\delta_n} \to x$ and $c^{\delta_n} \to c$ as $\delta_n \to 0$;
- $\overline{x^{\delta_n} c^{\delta_n}} \to \overline{xc}$ and $\overline{c^{\delta_n} x^{\delta_n}} \to \overline{cx}$ as $\delta_n \to 0$ as curves in metric (1.1).

In D^{δ_n}, we perform critical percolation on faces such that the faces near $\overline{c^{\delta_n} x^{\delta_n}}$ are colored black and those near $\overline{x^{\delta_n} c^{\delta_n}}$ are colored white. We define the exploration process γ^{δ_n} from x^{δ_n} to c^{δ_n} as follows: it starts from x^{δ_n} and is targeted at c^{δ_n}, and it makes turns so that its left side is black face. The path is uniquely determined by this rule. Our main conclusion is the following.

Theorem 1.1. *The exploration process γ^{δ_n} converges to SLE_6 in D from x to c in law in metric* (1.1)*.*

The strategy of the proof is as follows (we follow [4]):

1. Prove tightness to extract convergent subsequence by Prohorov's theorem (not today);
2. Define/parameterize the hull and identify its driving function.

We Assume Item 1 is done, i.e., we have extracted a convergent subsequence $\{\gamma^{\delta_n}\}$ that for some random continuous curve γ, we have $\gamma^{\delta_n} \to \gamma$ in law. By Skorohod's representation theorem, we can further assume $\gamma^{\delta_n} \to \gamma$ almost surely. We will prove γ is distributed like SLE_6 in D from x to c.

To avoid some technical difficulties, hereafter we assume $D = \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the unit disk. We parameterize $\gamma : [0, \infty) \to \overline{D}$ so that $\gamma(0) = x$ and $\lim_{t\to\infty} \gamma(t) = c$, and that it is not constant in any interval. We denote by K_t the hull formed by $\gamma[0,t]$ in $\overline{D}$ (i.e., closure in $\overline{D}$ of set of points disconnected by $\gamma[0,t]$ from c), and by D_t the connected component of $D\backslash\gamma[0,t]$ that has c on its boundary. Clearly, we have $K_t = \overline{D\backslash D_t}$. In order to relate γ with SLE_6, we have to map it onto upper half plane $\mathbb{H} := \{x+iy : y>0\}$ and measure it in upper half plane capacity. We fix a conformal map $\Phi : D \to \mathbb{H}$ that sends (x,c) to $(0,\infty)$.

1.1 Parameterize γ

For every $z \in D$, let σ_z (resp. $\sigma_z^{\delta_n}$) be the first time at which z is disconnected from c by γ in D (resp. from c^{δ_n} by γ^{δ_n} in D^{δ_n}).

Lemma 1.2. *Along some subsequence (still denoted by $\{\delta_n\}$), we have almost surely $\sigma_z^{\delta_n} \to \sigma_z$ for all $z \in D \cap \mathbb{Q}^2$.*

Proof. [1, Lemma 6.1, Lemma 6.2]. □

Hereafter, we consider the subsequence in Lemma 1.2 in place of the original one.

Lemma 1.3. *For all $u' < u$, there almost surely exists $v \in (u,u']$ such that $\gamma(v) \notin \gamma[0,u] \cup \partial D$.*

Proof. Suppose $\{x_j\}$ is a countable dense subset of in $\gamma[0,u] \cup \partial D$. By RSW estimate, we may argue that none of them are visited by $\gamma(u,u']$ almost surely. But γ is not constant in any interval. Therefore, the segment $\gamma(u,u']$ cannot always stay in $\gamma[0,u] \cup \partial D$. □

Lemma 1.4. *The process $t \mapsto K_t$ is strictly increasing almost surely.*

Proof. For $u' > t$, by Lemma 1.3, there exists $v \in (t,u')$ that $\gamma(v)$ is in one of the c.c.'s of $D\backslash\gamma[0,t]$. Suppose $\gamma(v)$ is not in D_t. Then by Lemma 1.2, there exists z (coordinantes are rational) that is in the same c.c. as $\gamma(v)$, and a.s. $\sigma_z^{\delta_n} \to \sigma_z$. Since $v > u \geqslant \sigma_z$ and $\gamma^{\delta_n} \to \gamma$, we know γ^{δ_n} will enter the c.c. it disconnected before. By definition of exploration process, it is not allowed. Therefore, we have $\gamma(v) \in D_t$. □

From these three lemmas, we may conclude that the capacity of $\{\Phi(K_t)\}_{t\geqslant 0}$ is strictly increasing and continuous in time. We parameterize γ by the capacity of $\{\Phi(K_t)\}_{t\geqslant 0}$, i.e., the upper half plane capacity of $\Phi(K_t)$ is t for each $t \in [0,\infty)$.

1.2 Identify a continuous martingale

Let us add two marked points $a \in \overline{cx}$, $b \in \overline{xc}$, and let $a^{\delta_n} \in \overline{c^{\delta_n}x^{\delta_n}}$, $b^{\delta_n} \in \overline{x^{\delta_n}c^{\delta_n}}$ be their approximations. Define two events:

$$\mathcal{A}^{\delta_n} := \left\{\gamma^{\delta_n} \text{ hits } \overline{c^{\delta_n}a^{\delta_n}} \text{ before } \overline{b^{\delta_n}c^{\delta_n}}\right\},$$

$$\mathcal{A} := \left\{\gamma \text{ hits } \overline{ca} \text{ before } \overline{bc}\right\}.$$

Lemma 1.2 tells us $\mathbb{P}[\mathcal{A}^{\delta_n}\Delta\mathcal{A}] \to 0$, where Δ means taking symmetric difference. Observe that $\mathcal{A}^{\delta_n}$ happens if and only if there exists a white crossing from $\overline{x^{\delta_n}b^{\delta_n}}$ to $\overline{c^{\delta_n}a^{\delta_n}}$.

Recall Cardy's formula: If $(\Omega_\delta; A_\delta, B_\delta, C_\delta, D_\delta) \to (\Omega; A, B, C, D)$ in the Carathéodory sence (i.e. for some conformal maps $\phi_\delta : \mathbb{D} = \{z : |z| < 1\} \to \Omega_\delta$ and $\phi : \mathbb{D} \to \Omega$, we have $\phi_\delta \to \phi$ locally uniformly, and $\phi_\delta^{-1}(X_\delta) \to \phi^{-1}(X)$ for $X \in \{A,B,C,D\}$), then

$$\mathbb{P}[A_\delta B_\delta \leftrightarrow C_\delta D_\delta] \to f(\Omega; A, B, C, D)$$

where f is conformally invariant and equals $|AB|/|CD|$ when Ω is an equilateral triangle with $\{A, C, D\}$ as its vertices.

Let us come back to our model. For all $t \geqslant 0$, on the one hand, if $\gamma[0,t] \cap \left(\overline{ca} \cup \overline{bc}\right) = \varnothing$, then there exists $N > 0$ (N may depend on γ) such that for all $n \geqslant N$, we have $\gamma^{\delta_n} \cap \left(\overline{c^{\delta_n} a^{\delta_n}} \cup \overline{b^{\delta_n} c^{\delta_n}}\right) = \varnothing$. Observe that now $\mathcal{A}^{\delta_n}$ happens if and only if there exists a white crossing from $\overline{\gamma^{\delta_n}(t), b^{\delta_n}}$ to $\overline{c^{\delta_n} a^{\delta_n}}$ in $D^{\delta_n} \backslash \gamma^{\delta_n}[0,t]$. By Cardy's formula, we have as $n \to \infty$,

$$\mathbb{E}[\mathbb{1}_{\mathcal{A}^{\delta_n}} | \gamma^{\delta_n}[0,t]] \to f(D_t; \gamma(t), b, c, a).$$

On the other hand, if $\gamma[0,t] \cap \left(\overline{ca} \cup \overline{bc}\right) \neq \varnothing$, then by Lemma 1.2, we have

$$\mathbb{E}[\mathbb{1}_{\mathcal{A}^{\delta_n}} | \gamma^{\delta_n}[0,t]] \to \mathbb{1}_{\mathcal{A}}$$

whose right hand side does not depend on t. Define

$$X_t := \begin{cases} f(D_t; \gamma(t), b, c, a) & \text{if } \gamma[0,t] \cap \left(\overline{ca} \cup \overline{bc}\right) = \varnothing, \\ \mathbb{1}_{\mathcal{A}} & \text{otherwise.} \end{cases}$$

Then, we have $\mathbb{E}[\mathbb{1}_{\mathcal{A}^{\delta_n}} | \gamma^{\delta_n}[0,t]] \to X_t$. We claim that X_t is a continuous martingale:

$$X_t = \mathbb{E}[\mathbb{1}_{\mathcal{A}} | \gamma[0,t]].$$

Indeed, for all bounded continuous function f on set of bounded curves (equipped with metric (1.1)), by Dominated Convergence Theorem and the fact $\mathbb{P}[\mathcal{A}^{\delta_n} \Delta \mathcal{A}] \to 0$, we have

$$\begin{aligned} \mathbb{E}[f(\gamma[0,t]) \mathbb{1}_{\mathcal{A}}] &= \lim_{n \to \infty} \mathbb{E}[\mathbb{1}_{\mathcal{A}^{\delta_n}} f(\gamma^{\delta_n}[0,t])] \\ &= \lim_{n \to \infty} \mathbb{E}\left[\mathbb{E}[\mathbb{1}_{\mathcal{A}^{\delta_n}} | \gamma^{\delta_n}[0,t]] f(\gamma^{\delta_n}[0,t])\right] \\ &= \mathbb{E}\left[X_t f(\gamma[0,t])\right]. \end{aligned}$$

By definition of X_t, we know X_t is adapted to filtration $\{\sigma(\gamma[0,t])\}_{t \geqslant 0}$ so we prove the claim by verifying the definition of conditional expectation. Hence X_t is a continuous martingale adapted to filtration of γ (although depends on a and b).

1.3 Derive the driving function

Recall that $\Phi : D \to \mathbb{H}$ is a fixed conformal map that sends (x, c) to $(0, \infty)$. For the two additional marked points a and b, we denote the their images by $a' := \Phi(a)$ and $b' := \Phi(b)$. Let $\{w_t\}_{t \geqslant 0}$ be the driving function of $\{\Phi(K_t)\}_{t \geqslant 0}$, and $\{g_t\}_{t \geqslant 0}$ be the solution of the following ODE:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - w_t}, \quad g_0(z) = z.$$

By the definition of driving function, we see that each g_t is a conformal map from $\mathbb{H} \backslash \Phi(K_t)$ onto $\mathbb{H}$ with the property that $\lim_{z \to \infty} |g_t(z) - z| = 0$.

Define

$$\Psi(z) := \frac{\int_0^z \frac{du}{u^{2/3}(1-u)^{2/3}}}{\int_0^1 \frac{du}{u^{2/3}(1-u)^{2/3}}}.$$

From Schwarz-Christoffle mapping theorem, we see that Ψ is the conformal map from the equilateral triangle $\tilde{\Delta}$ with vertices $\left(0,1,\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$ onto $\mathbb{H}$ sending $\left(0,1,\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$ to $(0,1,\infty)$. Define

$$R(z) := \frac{z-g_t(a')}{g_t(b')-g_t(a')},$$

then the composition $\Psi\circ R\circ g_t\circ\Phi$ is the conformal map from D_t onto $\tilde{\Delta}$ sending $(a,\gamma(t),b,c)$ to $(\tilde{a},\Phi(R(w_t)),\tilde{b},\tilde{c})$. Combining with Cardy's formula, we have

$$X_t = 1-\Psi(R(w_t)).$$

Note that the processes

$$\{w_t=\Psi^{-1}(1-X_t)(g_t(b')-g_t(a'))+g_t(a')\}_{t\geqslant 0},\quad \{g_t(a')\}_{t\geqslant 0},\quad \{g_t(b')\}_{t\geqslant 0}$$

are all adapted to filtration of γ. We write $w_t=M_t+V_t$ where V_t is of bounded variation and M_t is a local martingale. Denote by

$$Z_t := R(w_t) = \frac{w_t-g_t(a')}{g_t(b')-g_t(a')}.$$

Then

$$\begin{aligned}
d(1-X_t) &= d\Psi(Z_t) = d\left(\Psi\left(\frac{w_t-g_t(a')}{g_t(b')-g_t(a')}\right)\right)\\
&= \Psi'(Z_t)\left(\frac{dw_t}{g_t(b')-g_t(a')}+\frac{-(w_t-g_t(a'))}{(g_t(b')-g_t(a'))^2}\frac{2dt}{g_t(b')-w_t}+\frac{w_t-g_t(b')}{(g_t(b')-g_t(a'))^2}\frac{2dt}{g_t(a')-w_t}\right)\\
&\quad+\frac{1}{2}\Psi''(Z_t)\frac{d\langle M,M\rangle_t}{(g_t(b')-g_t(a'))^2}\\
&= \Psi'(Z_t)\frac{dw_t}{g_t(b')-g_t(a')}+\Psi'(Z_t)\left(\frac{1}{1-Z_t}-\frac{1}{Z_t}\right)\frac{-dt}{(g_t(b')-g_t(a'))^2}+\frac{1}{2}\Psi''(Z_t)\frac{d\langle M,M\rangle_t}{(g_t(b')-g_t(a'))^2}\\
&= \text{local martingale}+\Psi'(Z_t)\frac{dV_t}{g_t(b')-g_t(a')}+\frac{1}{2}\Psi''(Z_t)\frac{1}{(g_t(b')-g_t(a'))^2}\left(d\langle M,M\rangle_t-6dt\right)
\end{aligned}$$

where in the last equality we use $\Psi''(z)=\frac{2}{3}\left(\frac{1}{1-z}-\frac{1}{z}\right)\Psi'(z)$ for all $z\in\tilde{\Delta}$. Since X_t is a martingale, we get

$$\Psi'(Z_t)dV_t+\frac{1}{2}\Psi''(Z_t)\frac{d\langle M,M\rangle_t-6dt}{g_t(b')-g_t(a')}=0. \tag{1.2}$$

We choose a sequence $\{(a_n',b_n')\}$ so that $a_n'\to-\infty$ and $b_n'\to+\infty$, and that $Z_t^{(n)} := \frac{w_t-g_t(a_n')}{g_t(b_n')-g_t(a_n')}\in\left(\frac{5}{8},\frac{3}{4}\right)$. From Schwarz's Symmetric Principle, we have

$$\Psi'\left(Z_t^{(n)}\right)=\Theta(1),\quad \Psi''\left(Z_t^{(n)}\right)=\Theta(1).$$

Replacing a',b' by a_n',b_n' in (1.2) and taking $n\to\infty$, we have

$$dV_t=0. \tag{1.3}$$

Plugging into (1.2), we have

$$\Psi''(Z_t)\frac{d\langle M,M\rangle_t-6dt}{g_t(b')-g_t(a')}=0. \tag{1.4}$$

Replacing a',b' by a'_n,b'_n in (1.4), we have

$$d\langle M,M\rangle_t=6dt. \tag{1.5}$$

Equations (1.3) and (1.5) imply that w_t is $\sqrt{6}$ times a standard Brownian motion (and is started at $\Phi(x)=0$), hence γ is distributed like SLE_6. This finishes the proof.

2 Arm exponent: boundary case

We consider the critical percolation on faces of hexagonal lattice with mesh-size one. Define $B_{r,R}:=\{x\in\mathbb{C}:x=ae^{i\theta}\text{ for }r<a<R,0<\theta<\pi\}$. Let $A_{r,R}$ be the set of faces that intersect $B_{r,R}$. Let $p\geqslant 1$ be an integer and $\sigma\in\{\text{black},\text{white}\}^p$. We denote by $P^{\sigma}_{r,R}$ the probability that there exist p crossings $\{c_i\}_{i=1}^p$ in $A_{r,R}$ from the inner half circle to the outer half circle, arranged in clockwise order, such that c_i is in color σ_i.

The value $P^{\sigma}_{r,R}$ depends only on the cardinality of σ. Indeed, we can explore these crossings, one at a time, by starting exploration processes. By swtiching the colors in unexplored areas, we can switch the colors of these crossings. Denote by $P^{p}_{r,R}$ the value $P^{\sigma}_{r,R}$ where $\#\sigma=p$. Our main conclusion is the following.

Theorem 2.1. *As $R\to\infty$, we have*

$$\lim_{n\to\infty}P^p_{n,nR}=\Theta(1)R^{-\frac{p(p+1)}{6}}.$$

Remark 2.2. *We can estimate $P^p_{r,R}$ with r fixed: there exists $n_0>0$ that*

$$\lim_{R\to\infty}\frac{\log P^p_{n_0,R}}{\log R}=-\frac{p(p+1)}{6}.$$

This follows from Theorem 2.1 and some estimates (such as quasi-multiplicativity and extendability) for discrete model. See [3].

We simply denote the half annulus $B_{1,R}$ by B_R, whose four vertices are $(A_1,A_2,A_3,A_4):=(1,R,-R,-1)$. Recall a standard result in complex analysis: the π-extremal distance (i.e., π times the extremal distance) from $\overline{A_4A_1}$ to $\overline{A_2A_3}$ in B_R is $\log R$. We start a SLE_6 in B_R from A_4 to A_2 until it hits $\overline{A_1A_3}$. We denote by S the compact hull of this process, and by G the c.c. of $B_R\backslash S$ that has A_1 on its boundary, and by $\mathcal{L}$ the π-extremal distance from $\overline{A_4A_1}$ to $\overline{A_2A_3}$ in G. Note that if S hits $\overline{A_1A_2}$, then $\mathcal{L}=\infty$. To prove Theorem 2.1, we need the following theorem:

Theorem 2.3. *As $R\to\infty$, we have*

$$\mathbb{E}\left[\mathbb{1}_{\{\mathcal{L}<\infty\}}\exp(-\lambda\mathcal{L})\right]=\Theta(1)R^{-u(\lambda)}$$

where $\lambda\geqslant 0$ and $u(\lambda)=\frac{6\lambda+1+\sqrt{24\lambda+1}}{6}$.

2.1 Proof of Theorem 2.1

We will first construct a series of SLE_6's, and then use Theorem 2.3 together with convergence of exploration processes to derive the desired result. We construct these SLE_6 as follows.

- Let $Q^{(1)} := A_4$ and $D_1 := B_R$. We start a SLE_6 in D_1 from $Q^{(1)}$ to A_2 until it hits $\overline{A_1A_3}$. Denote by S_1 the hull of this SLE. If $S_1 \cap \overline{A_1A_2} \neq \varnothing$ then we stop the construction; otherwise, we let $Q^{(2)}$ be the point in $S_1 \cap \overline{A_4A_1}$ with least argument, and D_2 be the c.c. of $D_1 \backslash S_1$ that has A_1 on its boundary.

- Suppose for some $p \geqslant 2$, we have constructed $p-1$ SLE's and obtained starting point $Q^{(p)}$, domain D_p, and compact hull S_{p-1} that does not intersect $\overline{A_1A_2}$. We start a SLE_6 in D_p from $Q^{(p)}$ to A_2 until it hits $\overline{A_1A_3}$. Denote by S_p the hull of this SLE. If $S_p \cap \overline{A_1A_2} \neq \varnothing$ then we stop the construction; otherwise, we let $Q^{(p+1)}$ be the point in $S_p \cap \overline{A_4A_1}$ with least argument, and D_{p+1} be the c.c. of $D_p \backslash S_p$ that has A_1 on its boundary. We repeat this procedure until we stop somewhere.

Denote by $f_p(R)$ the probability that we can start at least $(p+1)$ SLE_6's in the above procedure. On the one hand, by convergence of exploration process to SLE_6 in Dobrushin domain, we have

$$\lim_{n\to\infty} P^p_{n,nR} = f_p(R).$$

On the other hand, by induction, we have

$$f_p(R) = \mathbb{E}\left[\mathbb{1}_{\{\mathcal{L}<\infty\}} f_{p-1}(\exp(\mathcal{L}))\right].$$

Theorem 2.3 with $\lambda = 0$ tells us that $f_1(R) = \mathbb{P}[\mathcal{L} < \infty] = \Theta(1)R^{-\frac{1}{3}}$. We will prove by induction that, for each $p \geqslant 1$, we have

$$f_p(R) = \Theta(1)R^{-\frac{p(p+1)}{6}}. \tag{2.1}$$

Suppose we have derived (2.1) for some $p \geqslant 1$. Then

$$\begin{aligned}
f_{p+1}(R) &= \mathbb{E}\left[\mathbb{1}_{\{\mathcal{L}<\infty\}} f_p(\exp(\mathcal{L}))\right] \\
&= \Theta(1)\mathbb{E}\left[\mathbb{1}_{\{\mathcal{L}<\infty\}} \exp\left(\frac{-p(p+1)}{6}\mathcal{L}\right)\right] \\
&= \Theta(1)R^{-u\left(\frac{p(p+1)}{6}\right)} && \text{(Use Theorem 2.3)} \\
&= \Theta(1)R^{-\frac{(p+1)(p+2)}{6}}.
\end{aligned}$$

This finishes the induction. Therefore, we obtain that for $p \geqslant 1$,

$$\lim_{n\to\infty} P^p_{n,nR} = \Theta(1)R^{-\frac{p(p+1)}{6}}.$$

2.2 Proof of Theorem 2.3

We follow [2] in this subsection. Let Φ be a conformal map from B_R onto upper half plane $\mathbb{H}$ that sends (A_1, A_2, A_3) to $(1, \infty, 0)$. Let $\tilde{x} = \Phi(A_4)$. A standard estimate in complex analysis tells us that, as $R \to \infty$, we have

$$R = \Theta(1)(1-\tilde{x})^{-1}.$$

We start a SLE_6 in $\mathbb{H}$ from $\tilde{x}$ to ∞. We denote by $\{K_t\}_{t\geqslant 0}$ the collection of its compact hulls, and by $\{W_t\}_{t\geqslant 0}$ its driving function (a Brownian motion started at $\tilde{x}$ with quadratic variation $6dt$). Let $\{g_t\}_{t\geqslant 0}$ be the solution of the following ODE:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_t(z) = z.$$

Then, each g_t is the conformal map from $\mathbb{H}\backslash K_t$ onto $\mathbb{H}$ with the property that $\lim_{z\to\infty} |g_t(z) - z| = 0$. Define

$$T_0 := \inf\{t : K_t \cap (-\infty, 0] \neq \varnothing \}, \quad T_1 := \inf\{t : K_t \cap [1,\infty) \neq \varnothing\}, \quad T = T_0 \wedge T_1,$$

and

$$f_t(z) := \frac{g_t(z) - g_t(0)}{g_t(1) - g_t(0)}.$$

Note that f_t is a renormalization of g_t that fixes points $(0, 1, \infty)$. We can prove that $f'_T(1) > 0$ if and only if $T_0 < T_1$, and that $T < \infty$ almost surely. See [2]. For $b \geqslant 0$ and $0 < x < 1$, we define

$$\Lambda(x, b) := \mathbb{E}_{1-x}\left[\mathbb{1}_{\{T_0<T_1\}} f'_T(1)^b\right],$$

$$\Omega(x, b) := \mathbb{E}_{1-x}\left[\mathbb{1}_{\{T_0<T_1\}} (1 - N_T)^b\right]$$

where $N_T = f_T(\max(K_T \cap \mathbb{R}))$, and $\mathbb{P}_y$ is the probability measure of SLE_6 in $\mathbb{H}$ that starts at y.

Recall that the hypergeometric function with index (α, β, γ) is defined as

$${}_2F_1(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{z^n}{n!},$$

where $(u)_0 := 1$ and $(u)_n := u(u + 1) \ldots (u + n - 1)$ for $n \geqslant 1$. The function ${}_2F_1(\alpha, \beta, \gamma; \cdot)$ solves

$$(x - 1)xF''(x) + (\gamma - (\alpha + \beta + 1)x)F'(x) - \alpha\beta F(x) = 0.$$

To prove Theorem 2.3, we need the following two lemmas.

Lemma 2.4. *We have*

$$\Lambda(x, b) = \frac{\sqrt{\pi}2^{-2\hat{b}}\Gamma(5/6 + \hat{b})}{\Gamma(1/3)\Gamma(1 + \hat{b})} x^{1/6+\hat{b}} {}_2F_1(1/6 + \hat{b}, 1/2 + \hat{b}, 1 + 2\hat{b}; x)$$

where $\hat{b} = \frac{\sqrt{24b+1}}{6}$, *and* ${}_2F_1$ *is the hypergeometric function.*

Lemma 2.5. *For all* $x \in (0, 1)$, *we have*

$$\left(\frac{x}{2}\right)^b \Lambda(x/2, b) \leqslant \Omega(x, b) \leqslant x^b \Lambda(x, b).$$

Proof of Theorem 2.3 (assuming these two lemmas). Denote by $\mathcal{L}^*$ the π-extremal distance from $(-\infty, 0]$ to $(N_T, 1)$. Conformal invariance of SLE gives that $\mathcal{L} \overset{(d)}{=} \mathcal{L}^*$. We have a standard estimate in complex analysis:

$$\mathcal{L}^* = -\log(1 - N_T) + O(1)$$

where $O(1)$ is a quantity with bound that does not depend on N_T. Then,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{\{\mathcal{L}<\infty\}}\exp(-\lambda\mathcal{L})\right] &= \mathbb{E}_{\tilde{x}}\left[\mathbb{1}_{\{\mathcal{L}^*<\infty\}}\exp(-\lambda\mathcal{L}^*)\right] \\
&= \Theta(1)\mathbb{E}_{\tilde{x}}\left[\mathbb{1}_{\{T_0<T_1\}}(1-N_T)^{\lambda}\right] \\
&= \Theta(1)\Omega(1-\tilde{x}, \lambda) \\
&= \Theta(1)(1-\tilde{x})^{\lambda}(1-\tilde{x})^{\frac{1+\sqrt{1+24\lambda}}{6}} \qquad \text{(By Lemma 2.4 and Lemma 2.5)} \\
&= \Theta(1)(1-\tilde{x})^{u(\lambda)} \\
&= \Theta(1)R^{-u(\lambda)}.
\end{aligned}$$

This finishes the proof. □

We now prove Lemma 2.4.

Proof of Lemma 2.4. Recall that $f_t(z) = \frac{g_t(z)-g_t(0)}{g_t(1)-g_t(0)}$. We define

$$Z_t = \frac{W_t - g_t(0)}{g_t(1) - g_t(0)}$$

and time-change

$$s = s(t) = \int_0^t \frac{du}{(g_u(1) - g_u(0))^2}, \qquad \text{for } t < T.$$

Since $T < \infty$ almost surely, we have $s_0 := \lim_{t \to T-} s(t) < \infty$ almost surely. Define

$$\tilde{Z}(s) := Z(t(s)), \quad \tilde{f}_s(z) := f_{t(s)}(z), \quad \alpha(s) := \log \tilde{f}_s'(1).$$

Then, by using Ito's formula, we can derive

$$d\tilde{Z}_s = dX_s + \left(\frac{2}{\tilde{Z}(s)} - \frac{2}{1 - \tilde{Z}(s)}\right) ds,$$

$$\partial_s\left(\alpha(s)\right) = \frac{-2}{(\tilde{Z}(s) - 1)^2} - \frac{2}{\tilde{Z}(s)} - \frac{2}{1 - \tilde{Z}(s)}$$

where X is a Brownian motion with quadratic variation $6ds$. Hence $(\tilde{Z}(s), \alpha(s))$ is a continuous Markov process. Define

$$y(x, v) = \mathbb{E}\left[\exp(b\alpha(s_0)) | \tilde{Z}(0) = x, \alpha(0) = v\right].$$

Then we have the following facts:

- The process $Y_s := y\left(\tilde{Z}(s), \alpha(s)\right)$ is a local martingale (by the Strong Markov Property). The drift term in Ito's formula of dY vanishes, which gives

$$\left(\frac{2}{x} - \frac{2}{1-x}\right)\partial_x y + 3\partial_{xx} y + \left(\frac{-2}{(x-1)^2} + \frac{-2}{x} + \frac{-2}{1-x}\right)\partial_v y = 0;$$

- We have that $y(x,v) = \exp(bv)y(x,0)$ and that $y(1-x,0) = \Lambda(x,b) =: h(x)$. Therefore,

$$y(x,v) = \exp(bv)h(1-x).$$

Combining these two facts, we have

$$-2bh(x) + 2x(1-2x)h'(x) + 3x^2(1-x)h''(x) = 0. \tag{2.2}$$

We solve (2.2) by looking for solutions of type

$$h(x) = x^c z(x)$$

where c is some constant. After discarding a solution that blows up at 0, we know $\Lambda(x,b) = h(x)$ is of the desired form. □

Proof of Lemma 2.5. [2, Lemma 3.3]. □

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