Combinatorics, Lecture 7, 2022/5/31

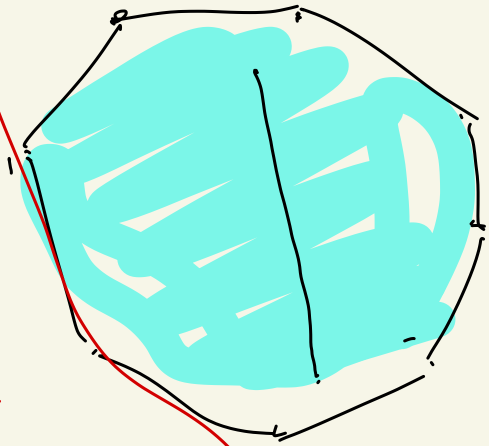
Let $K_n^{(r)}$ be the complete r-uniform hypergraph on n vertices, i.e. $K_n^{(r)} = (V, \binom{V}{r})$ with $|V| = n$.

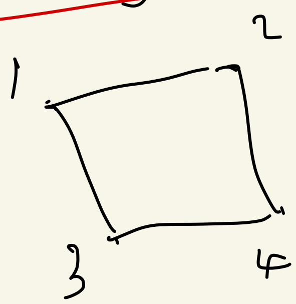
Def. The hypergraph Ramsey number $R^{(r)}(s,t)$ is the least integer N such that any 2-edge-coloring of $K_N^{(r)}$ has a blue $K_s^{(r)}$ or a red $K_t^{(r)}$.

Ex: $\forall\, s,t \geq r$, $R^{(r)}(s,t) < +\infty$.

Erdős-Szekeres Thm For any integer n, there exists an integer $N(n)$ such that any collection of $N \geq N(n)$ points in the plane, no three on a line, has a subset of n points forming a convex n-gon. ✓

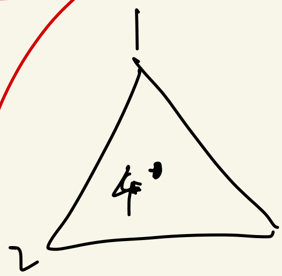
pf: We observe that n points forms a convex n-gon iff every quadrilateral formed by taking 4 points is convex.





convex

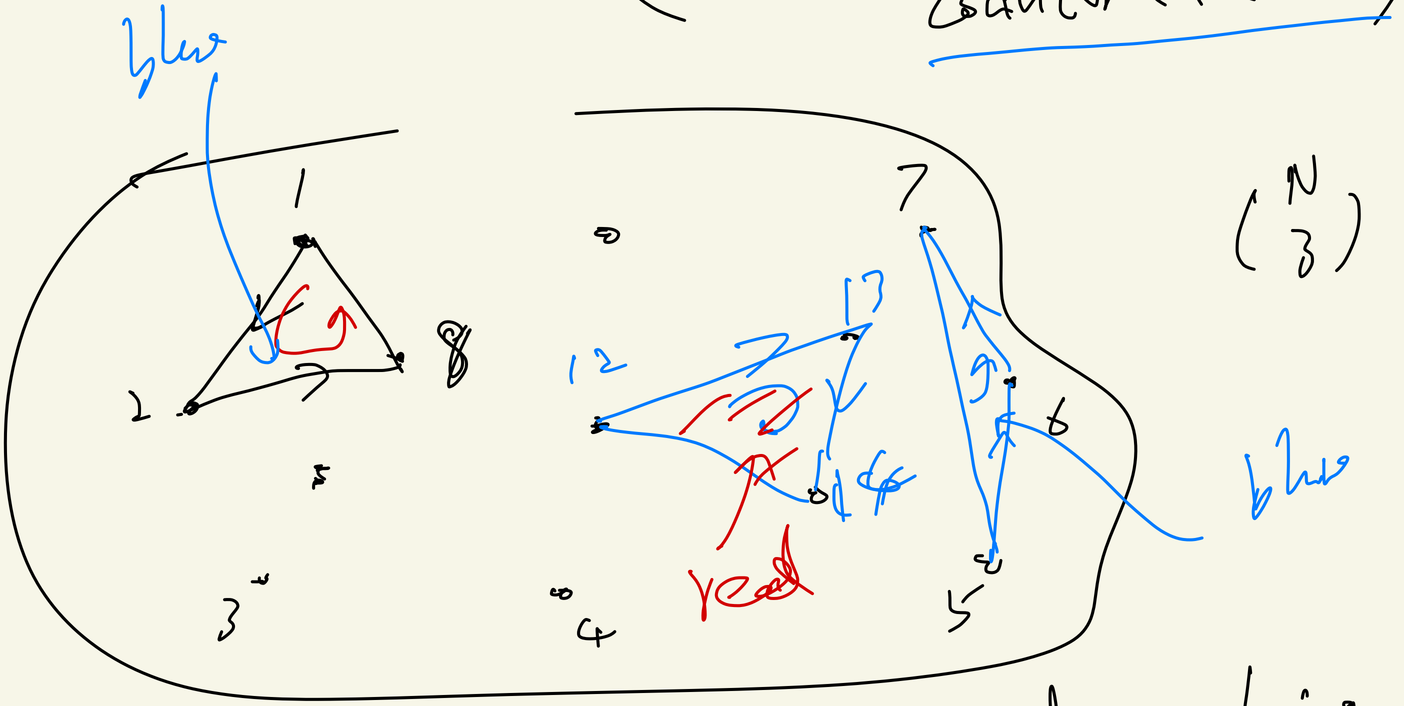
or



non-convex

Let $N(n) = R^{(3)}(n,n)$. We will show that it is enough to have $N(n)$ points in the plane.

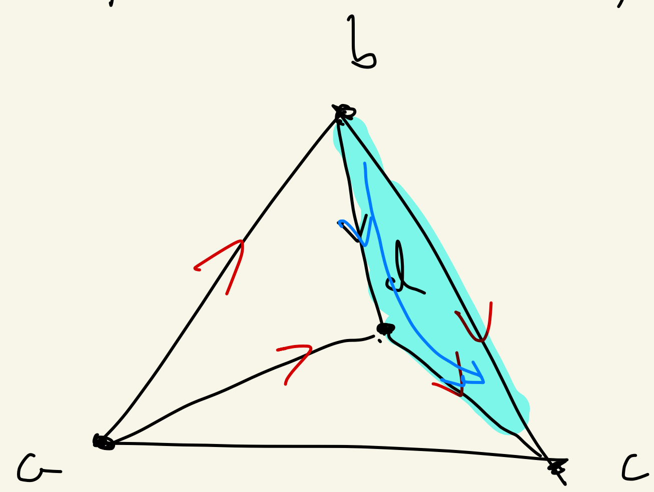
Let S be a set of $N(n)$ such points. Number the points arbitrarily and color any triangle (every 3-subset) red if the path from the smallest number via the middle one to the largest number is clockwise (color it blue if it is counter clockwise)



In this way, we get a 2-edge-coloring of $K_N^{(3)}$. By Ramsey's number, we have a monochromatic (red or blue) $K_n^{(3)}$. We may assume it is red.

Claim. We shall prove that this red $K_n^{(3)}$ gives an convex n-gon. So it is enough to prove that there are

NO four points giving the following configuration.



Let $a<b<c$

Consider adc, red $\Rightarrow a<d<c$

Consider abd, red $\Rightarrow a<b<d$

$\Longrightarrow$ $a<b<d<c$

But bdc is blue!

This contradiction completes the proof. ☒

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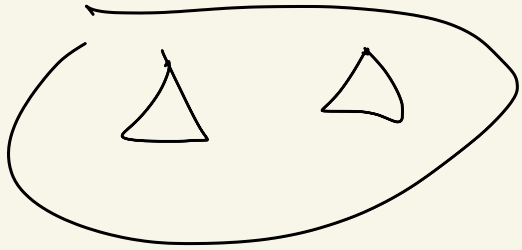
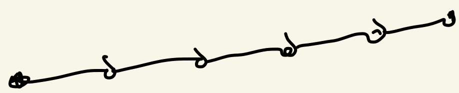
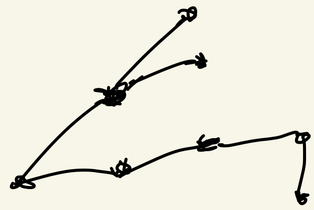
§ Trees

Def. A graph G is connected, if for any two vertices $u, v \in V(G)$, there is a path from u to v.

i.e  disconnected

Def. A graph is a <u>tree</u> if it is <u>connected</u> but has <u>NO cycle</u>.

i.e.  

A vertex is called a <u>leaf</u> in a tree if its degree is 1.

planar G
n vtx, e edges
f faces
$\Rightarrow n-e+f=2$

Euler's formula on tree

For any tree $T=(V,E)$, we have
$$|V|=|E|+1.$$

Pf. By induction. (Each tree has a leaf) $\square$

Fact. Each tree with at least 2 vertices has at least 2 leaves

<u>Thm 1</u> (<u>Tree characterization</u>) Let $T=(V,E)$ be a graph. Then the following are equivalent.

(1) T is a tree (i.e., connected and NO cycle)

(2) T is a <u>minimal</u> connected graph (i.e., deleting any edge will result in a disconnected graph)

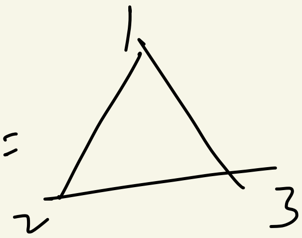
(3) T is a <u>maximal</u> graph without a cycle (i.e., adding any new edge will result in a cycle.)

<u>Pf</u>: <u>exercise</u>. $\boxtimes$

<u>Def</u>. Given a graph G, a subgraph H of G is a <u>spanning subgraph</u> if $V(H)=V(G)$.

<u>Fact 2</u>. G is connected iff it contains a spanning tree.

<u>Def</u>. For a connected graph G with n vertices $v_1, v_2, \cdots, v_n$, let $ST(G)$ be the number of <u>labelled spanning trees in G</u>

i.e. $G =$ 

$ST(G) = 3$

<u>Thm 2</u> (Cayley's formula) For $n \geq 2$,

$$ST(K_n) = n^{n-2}$$

We will give two proofs for this.

Given a spanning tree T in G,

$$\sum_{i=1}^{n} d_T(v_i) = 2e(T) = 2(n-1)$$

<u>Lemma 1</u>. Let $d_1, d_2, \cdots, d_n$ be positive integers with $\sum_{i=1}^{n} d_i = 2n-2$. Then the number of spanning trees T in K_n on the vertex set

$\{v_1, v_2, \dots, v_n\}$ satisfying $d_T(v_i) = d_i$ is

equal to $\dfrac{(n-2)!}{(d_1-1)!\,(d_2-1)!\cdots(d_n-1)!}$

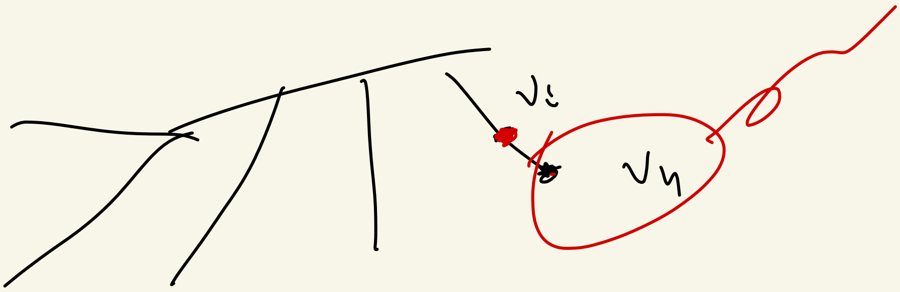
Pf: We prove by induction n.

The base case is trivial, i.e. $n=2$. ✓

Now we may assume that this statement holds for any sequence of $n-1$ positive integers whose sum equals $2(n-1)-2 = 2n-4$.

Consider $d_1, \dots, d_n$ with $\sum_i d_i = 2n-2$.

By average, there exists some $d_i = 1$, say $d_n = 1$.



Let $\mathcal{F}_i$ be the family of all trees T with $d_T(v_j) = d_j$ $\forall$ $j \in [n]$.

For $1 \leq i \leq n-1$, let

$$\mathcal{T}_i = \{ T - v_n : T \in \mathcal{T} \text{ and the unique neighbor of } v_n \text{ in } T \text{ is } v_i \}$$

$$\Rightarrow \quad |\mathcal{T}| = \sum_{i=1}^{n-1} |\mathcal{T}_i| ,$$ and all

trees in $\mathcal{T}_i$ have $n-1$ vertices $v_1, \dots, v_{n-1}$

such that $\begin{cases} d(v_j) = d_j & \text{if } j \neq i \\ d(v_i) = d_i - 1 . \end{cases}$

By induction, we have

$$|\mathcal{T}_i| = \frac{(n-3)!}{(d_1-1)! \cdots (d_i-2)! \cdots (d_{n-1}-1)!}$$

$$= \frac{(n-3)! \cdot (d_i - 1)}{(d_1-1)! \cdots (d_{n-1}-1)!}$$

$$\Rightarrow \quad |\mathcal{T}| = \sum_{i=1}^{n-1} |\mathcal{T}_i|$$

$$= \frac{(n-3)!}{(d_1-1)! \cdots (d_{n-1}-1)!} \quad \boxed{\sum_{i=1}^{n-1} (d_i - 1)} = n-2$$

$$= \frac{(n-2)!}{(d_1-1)! \cdots (d_n-1)!} \quad (\text{as } d_n = 1)$$

$\boxtimes$

Proof of Cayley's formula (1st proof).

Recall that Multinomial Theorem:

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{i_1 + \cdots + i_k = n \\ i_j \geq 0}} \frac{n!}{i_1! \cdots i_k!} x_1^{i_1} \cdots x_k^{i_k}.$$

$\Rightarrow$ (let $x_1 = \cdots = x_k = 1$)

$$\boxed{k^n = \sum_{\substack{i_1 + \cdots + i_k = n \\ i_j \geq 0}} \frac{n!}{i_1!\, i_2! \cdots i_k!}}$$

So we have from the Lemma,

$$ST(K_n) = \sum_{\substack{\sum_{i=1}^{n} d_i = 2n-2 \\ d_i \geq 1}} \frac{(n-2)!}{(d_1-1)! \cdots (d_n-1)!}$$

$x_i = d_i - 1$

$$= \sum_{\substack{\sum_{i=1}^{n} x_i = n-2 \\ x_i \geq 0}} \frac{(n-2)!}{x_1!\, x_2! \cdots x_n!}$$

$$= n^{n-2}.$$

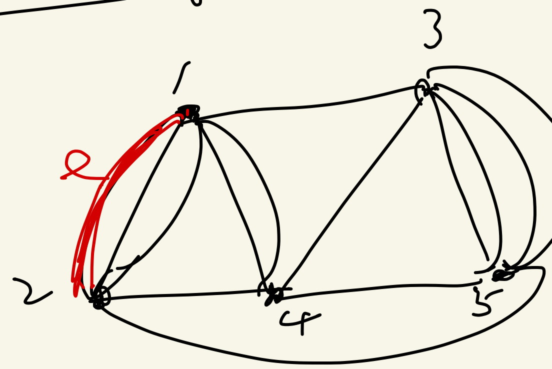
To start the 2^{nd} proof, we need some notion.

Def. A multi-graph is a loopless graph, where we allow multiple edges between 2 vertices.

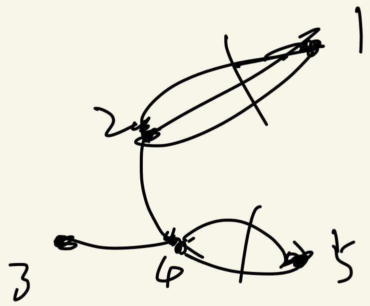
Def. For a multigraph G on $[n]$, we define the Laplace matrix $Q = (q_{ij})_{n \times n}$ of G as follows:

$$q_{ij} = \begin{cases} d_G(i), & \text{if } i = j \\ -m, & \text{if } i \neq j \text{ and there are } m \text{ edges between } i \text{ and } j. \end{cases}$$

For example.



$$Q = \begin{bmatrix} 6 & -3 & -1 & -2 & 0 \\ -3 & 5 & 0 & -1 & -1 \\ -1 & 0 & 6 & -1 & -4 \\ -2 & -1 & -1 & 5 & -1 \\ 0 & -1 & -4 & -1 & 6 \end{bmatrix}$$



$\Rightarrow$ ST = 6

Def For $n \times n$ matrix Q, let Q_{ij} be the $(n-1) \times (n-1)$ matrix by deleting the i^{th} row and the j^{th} column.

i.e. $$Q_{11} = \begin{bmatrix} 5 & 0 & -1 & -1 \\ 0 & 6 & -1 & -4 \\ -1 & -1 & 5 & -1 \\ -1 & -4 & -1 & 6 \end{bmatrix}$$

Prop. For $\forall\ 1 \le i, j \le n$, (for Laplace matrix Q) $\det(Q_{ij})$ are of the same value.

Thm 3. For any multigraph G,

$ST(G) = \det(Q_{11})$, where Q is the Laplace matrix of G.

Pf. We observe that if G has an isolated vertex v_1 (i.e. $d_G(v_1) = 0$), then $ST(G) = 0$. Also $\det(Q_{11}) = 0$, because the 1^{st} column of Q consists of zeros, which implies that the sum of the rows of Q_{11} is also zero and thus Q_{11} has an eigenvalue 0.

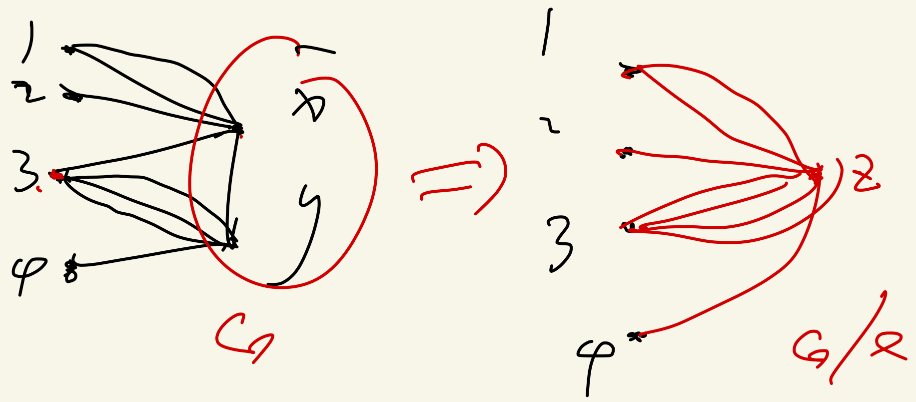
We prove by induction on $e(G)$. The base case is trivial when $e(G) = 1$. Now we may assume the statement holds for any multigraph with less than $e(G)$ edges.

Fix an edge $e = xy$.

$\Rightarrow$ Define two multigraphs $e = \Big|\begin{smallmatrix} x \\ y \end{smallmatrix}$

(1) $G - e$,

(2) G/e is the multigraph obtained from G by contracting x and y into a new vertex z and adding all new edges in $\{zu : xu \in E(G)\} \cup \{zu : yu \in E(G)\}$



Let Q' and Q'' be the Laplace matrices of $G-e$ and G/e, respectively.

As G has no isolated vertex. So we may assume the edge e has $1, 2$ as its endpoints.

i.e.

$$G-e \Rightarrow Q' = \begin{bmatrix} \boxed{\begin{matrix} 5 & -2 \\ -2 & 4 \end{matrix}} & \begin{matrix} -1 & -2 & 0 \\ 0 & -1 & -1 \end{matrix} \\ \begin{matrix} -1 & 0 \\ -2 & -1 \\ 0 & -1 \end{matrix} & \boxed{\begin{matrix} 6 & -1 & -4 \\ -1 & 5 & -1 \\ -4 & -1 & 6 \end{matrix}} \end{bmatrix}$$

$$G/e \Rightarrow Q'' = \begin{bmatrix} 5 & \begin{matrix} -1 & -3 & -1 \end{matrix} \\ \begin{matrix} -1 \\ -3 \\ -1 \end{matrix} & \boxed{\begin{matrix} 6 & -1 & -4 \\ -1 & 5 & -1 \\ -4 & -1 & 6 \end{matrix}} \end{bmatrix}$$

Let $Q_{11,22}$ be obtained from Q by deleting the first 2 rows and the first 2 columns. Then we have

$$\det(Q_{11}) = \det\left((Q')_{11}\right) + \det(Q_{11,22}).$$

✓

And $Q_{11,22} = (Q'')_{11}$

$$\Rightarrow \det(Q_{11}) = \det\left((Q')_{11}\right) + \det\left((Q'')_{11}\right).$$

8

✓

Claim For any edge e, we have

$$ST(G) = ST(G-e) + ST(G/e) \quad \checkmark$$

Pf. We can divide the spanning trees of G into two classes

① The 1st class containing these spanning trees in G NOT containing e; $\Leftrightarrow$ $ST(G-e)$

② The 2nd - - - - - - containing e. $\Leftrightarrow$ $ST(G/e)$

By induction, we have

$$ST(G-e) = \det((Q')_{11})$$

and

$$ST(G/e) = \det((Q'')_{11}).$$

Putting all above together, we have

$$ST(G) = ST(G-e) + ST(G/e)$$

$$= \det\left((Q')_{11}\right) + \det\left((Q'')_{11}\right)$$

$$= \det(Q_{11}) \quad \blacksquare$$

$\boxed{\text{2nd proof of Cayley's formula}}$.

Let $G = K_n$. Then

$$Q = \begin{pmatrix} n-1 & & -1 \\ & \ddots & \\ -1 & & n-1 \end{pmatrix}.$$

$$\Longrightarrow ST(K_n) = \det(Q_{11}) = n^{n-2}. \quad \blacksquare$$

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§3. System of distinct representative (SDR)

Def. A system of distinct representative (SDR) for a sequence of sets $S_1, \ldots, S_m$ (可以相同) is a sequence of distinct elements $x_1, \ldots, x_m$ such that $x_i \in S_i$ for each $i \in [m]$.

Fact (*). if $S_1, S_2, \ldots, S_m$ have a SDR, then the union of any k sets has at least k elements, i.e.

$$\left| \bigcup_{i \in I} S_i \right| \geq |I| \quad \text{for any } I \subseteq [m].$$

Hall's condition

Thm (Hall's Thm) The sets $S_1, \ldots, S_m$ have a SDR if and only if $S_1, \ldots, S_m$ satisfy Hall's condition.

pf: . ——— ☒