Geometry of Special Nilpotent Orbits

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Nilpotent matrices

- M_n = the set of $n \times n$ matrices.

Proposition

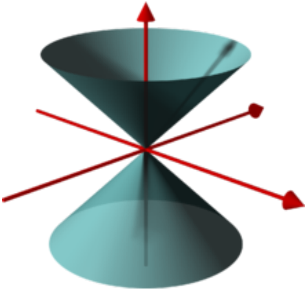
The following are equivalent:

- $A \in M_n$ *is nilpotent, i.e.* $A^k = 0$ *for some* k.
- $A^n = 0$.
- $\det(\lambda I - A) = \lambda^n$.
- $\mathrm{tr}(A) = \mathrm{tr}(A^2) = \cdots = \mathrm{tr}(A^n) = 0$.
- A *is conjugate to* $\mathrm{diag}(J_{d_1}, \cdots, J_{d_l})$ *with* $\sum_i d_i = n$.

$$J_d = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \cdots & & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

2×2 nilpotent matrices

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is nilpotent iff $\mathrm{tr}(A) = \mathrm{tr}(A^2) = 0$, which is equivalent to $a + d = 0$ and $a^2 + bc = 0$.



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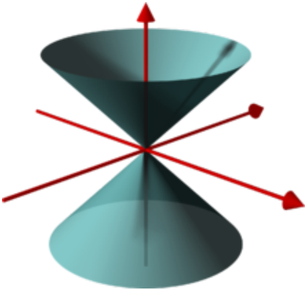

Any nilpotent matrice is conjugate to one of the following:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Nilpotent orbits and partitions

Nilpotent matrices in M_4 are conjugate to one of the following:

$$\begin{bmatrix} \circ & 1 & \circ & \circ \\ \circ & \circ & 1 & \circ \\ \circ & \circ & \circ & 1 \\ \circ & \circ & \circ & \circ \end{bmatrix} \quad \begin{bmatrix} \circ & 1 & \circ & \circ \\ \circ & \circ & 1 & \circ \\ \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \end{bmatrix} \quad \begin{bmatrix} \circ & 1 & \circ & \circ \\ \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & 1 \\ \circ & \circ & \circ & \circ \end{bmatrix} \quad \begin{bmatrix} \circ & 1 & \circ & \circ \\ \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \end{bmatrix} \quad \begin{bmatrix} \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \end{bmatrix}$$

$[4], \quad [3,1], \quad [2,2], \quad [2,1,1], \quad [1,1,1,1].$

Nilpotent orbits and partitions

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$[4], \quad [3,1], \quad [2,2], \quad [2,1,1], \quad [1,1,1,1].$

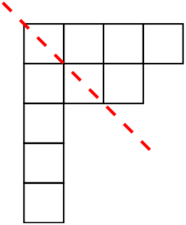
Proposition

Conjugacy classes of nilpotent matrices in M_n (called nilpotent orbits in M_n) are in bijection with partitions of n.

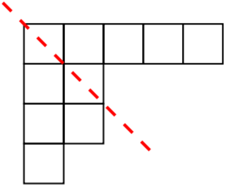
$$\begin{pmatrix} J_{d_1} & 0 & 0 & \cdots & 0 \\ 0 & J_{d_2} & 0 & \cdots & 0 \\ & \cdots & & \cdots & \\ 0 & 0 & 0 & 0 & J_{d_k} \end{pmatrix} \leftrightarrow \mathbf{d} = [d_1, \cdots, d_k].$$

$d_1 \geq d_2 \geq \cdots \geq d_k, \quad d_1 + \cdots + d_k = n.$

Partitions, Young diagram and duality



$[4, 3, 1, 1, 1]$

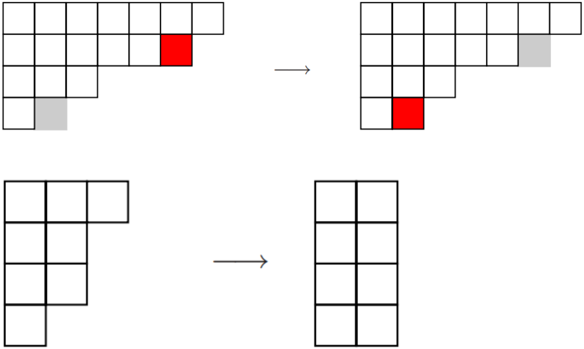


$[5, 2, 2, 1]$

$\mathbf{d} = [d_1, \cdots, d_k]$

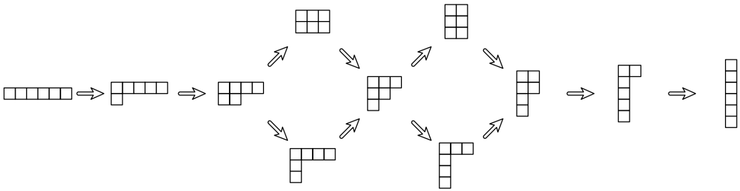
dual partition $\mathbf{e} = [e_1, \cdots, e_m]$
$e_j = \sharp\{i | d_i \geq j\}$.

Degenerations of Young diagrams

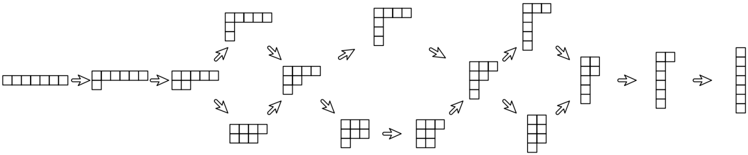


In terms of partition: **d** is a degeneration of **e**, denoted by $\mathbf{d} \leq \mathbf{e}$, if for all k, we have $d_1 + \cdots + d_k \leq e_1 + \cdots + e_k$.

More degenerations



Nilpotent orbits in $\mathfrak{sl}_6$



Nilpotent orbits in $\mathfrak{sl}_7$

Closure order

Denote by $\mathcal{O}_{\mathbf{d}}$ the nilpotent orbit corresponds to the partition **d**, by $\overline{\mathcal{O}}_{\mathbf{d}}$ its closure in M_n.

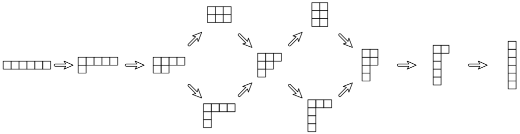
Proposition

$\mathcal{O}_{\mathbf{d}} \subset \overline{\mathcal{O}}_{\mathbf{e}}$ *if and only if* $\mathbf{d} \leq \mathbf{e}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\lim_{t\to 0}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus the closure of $\mathcal{O}_{[4]}$ contains $\mathcal{O}_{[2,2]}$ (also $\mathcal{O}_{[3,1]}$ in a similar way)

Hasse diagram



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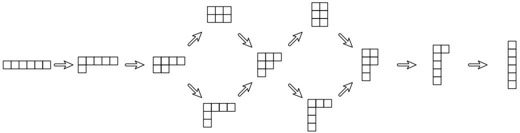


$[6]$ — 30

$[5,1]$ — 28

$[4,2]$ — 26

$[4,1^2]$ $[3^2]$ — 24

$[3,2,1]$ — 22

$[3,1^3]$ $[2^3]$ — 18

$[2^2,1^2]$ — 16

$[2,1^4]$ — 10

$[1^6]$ — 0

Symmetric via the dual partitions!

Simple Lie algebras

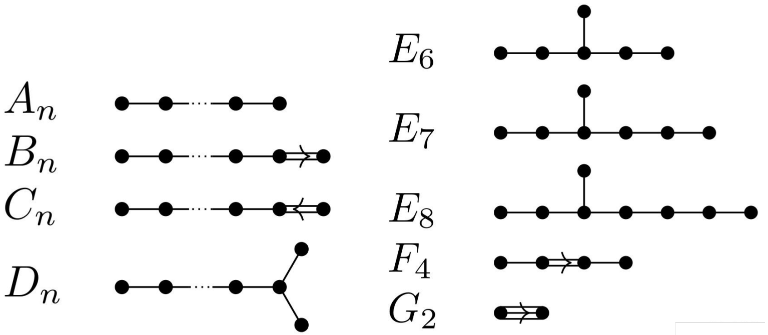
Theorem (W. Killing (1847-1923), E. Cartan (1869-1951))

classification of complex simple Lie algebras.

- $A_\ell = \mathfrak{sl}_{\ell+1} = \{A \in M_{\ell+1} | \mathrm{tr}(A) = 0\}$.
- $B_\ell = \mathfrak{so}_{2\ell+1} = \{A \in M_{2\ell+1} | A + A^t = 0\}$.
- $C_\ell = \mathfrak{sp}_{2\ell} = \{A = \begin{pmatrix} A_1 & A_2 \\ A_3 & -A_1^t \end{pmatrix} \in M_{2\ell} | A_2 = A_2^t, A_3 = A_3^t\}$.
- $D_\ell = \mathfrak{so}_{2\ell} = \{A \in M_{2\ell} | A + A^t = 0\}$.
- Exceptional types:

$$G_2, F_4, E_6, E_7, E_8$$

Dynkin diagram



Dynkin made major contributions to the theory of Lie algebras (Coxeter-Dynkin diagrams) and to probability (Markov process).

Simply-laced vs non simply-laced

Simply-laced: A, D, E Non simply-laced: B, C, F, G

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Simply-laced vs non simply-laced

Simply-laced: A, D, E Non simply-laced: B, C, F, G

Cartan's doctoral thesis of 1894 contains a major contribution to Lie algebras where he completed the classification of the semi-simple algebras over the complex field which Killing had essentially found. However, although Killing had shown that only certain exceptional simple algebras were possible, he had not proved that in fact these algebras exist. This was shown by Cartan in his thesis when he constructed each of the exceptional simple Lie algebras over the complex field.

Nilpotent orbits in general

Given any semisimple Lie algebra $\mathfrak{g}$ and G its adjoint Lie group.

- $x \in \mathfrak{g}$ is **nilpotent** if $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}, y \mapsto [x, y]$ is a nilpotent linear map.
- $x \in \mathfrak{g}$ nilpotent, its G-orbit $\mathcal{O}_x := G \cdot x$ is called a nilpotent orbit.
- There are only finitely many nilp. orbits in $\mathfrak{g}$, whose union is the **nilpotent cone** $\mathcal{N}$, which is normal and a complete intersection (Kostant).
- Nilpotent orbits can be parametrized by weighted Dynkin diagrams or by partitions (for classical cases).

Nilpotent orbit closures

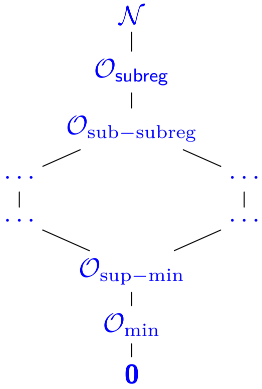
Let $\overline{\mathcal{O}_x} \subset \mathfrak{g}$ be the closure of a nilpotent orbit. Then:

- $\overline{\mathcal{O}_x}$ consists of $\mathcal{O}_x$ and smaller nilpotent orbits, hence it is smooth in codimension 1.
- $\mathcal{O}_x$ is stable under the dilation action of $\mathbb{C}^*$, hence $\overline{\mathcal{O}_x}$ is cone-like and $0 \in \overline{\mathcal{O}_x}$.
- The symplectic structure satisfies $\lambda^*\omega = \lambda\omega$, $\forall \lambda \in \mathbb{C}^*$.

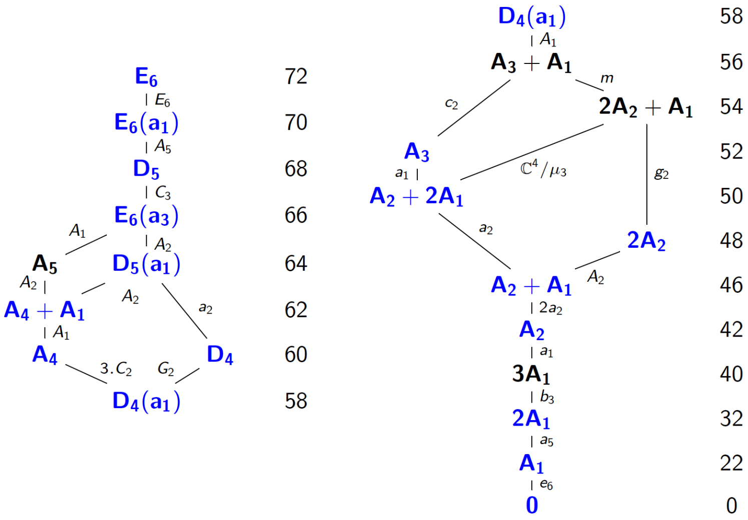
But in general, $\overline{\mathcal{O}_x}$ is not normal:

- $\overline{\mathcal{O}_x} \subset \mathfrak{sl}_n$ is always normal.
- $\overline{\mathcal{O}}_{min}$ is normal with an isolated singularity.
- Orbits with normal closure are known for classical cases (Kraft-Procesi), G_2 (Kraft), F_4(Broer), E_6 (Broer, Sommers).

Hasse diagram



An example of Hasse diagram: E_6 case



Why nilpotent orbits?

Beauville introduced symplectic singularities via the analogy:

Calabi-Yau

smooth variety with a nowhere vanishing volume form

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normal singularities whose smooth locus admits a nowhere vanishing volume form *which extends to any resolution*.

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HyperKähler manifolds

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Symplectic singularities

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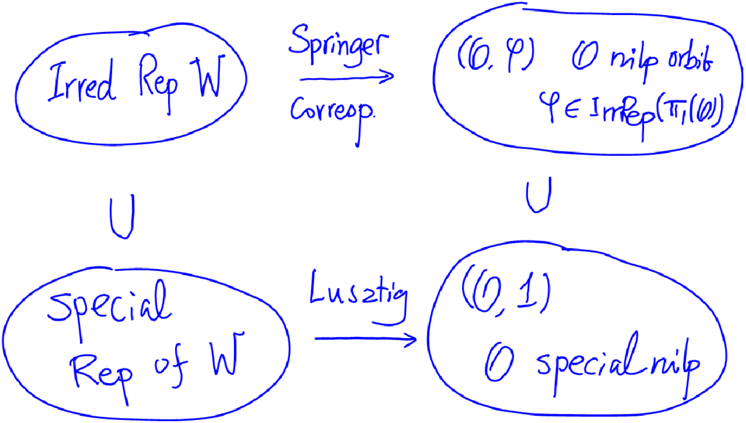
Nilpotent orbit closures form important classes of symplectic sing..

Springer correspondence and special orbits



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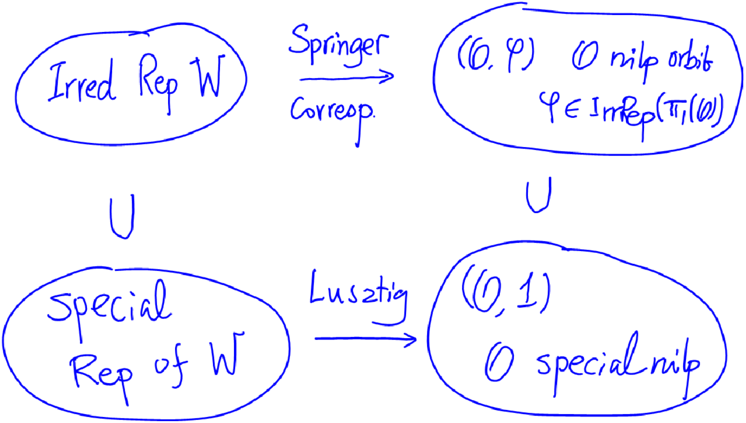


Definition

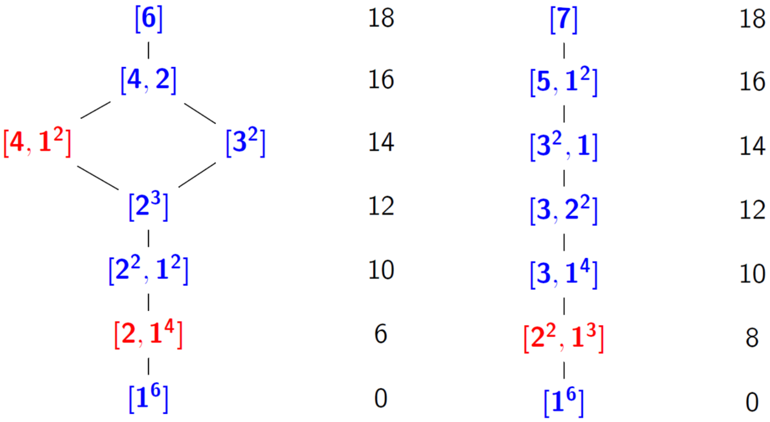
A nilpotent orbit $\mathcal{O}$ is called special if $(\mathcal{O}, 1)$ corresponds to a special representation of the Weyl group.

Geometry of special nilpotent orbits: motivation

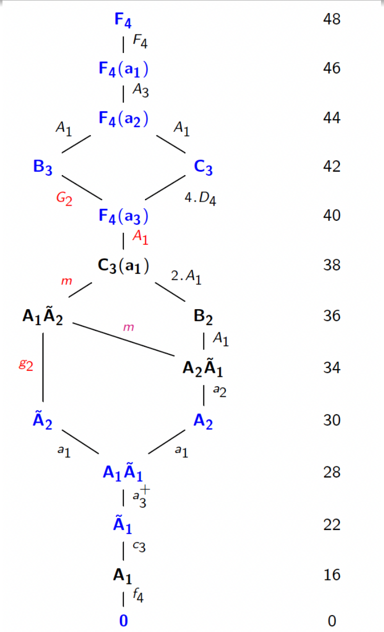
Quotes from Lusztig's paper "Notes on unipotent classes" (Asian J. Math 1997)

The special unipotent classes play a key role in several problems in representation theory, such as the classification of irreducible complex representations of a reductive group over a finite field, and the classification of primitive ideals in the enveloping algebra of a semisimple Lie algebra. Unfortunately, their definition is totally ungeometrical. For this reason, special unipotent classes are often regarded as rather mysterious objects. To partially remedy this situation, we have felt the need to try to unveil some of the purely geometrical properties of special unipotent classes, or rather, of the closely connected special pieces (defined below); this has led to the present paper.

Special orbits in $\mathfrak{sp}_6$ and $\mathfrak{so}_7$



Special orbits in F_4



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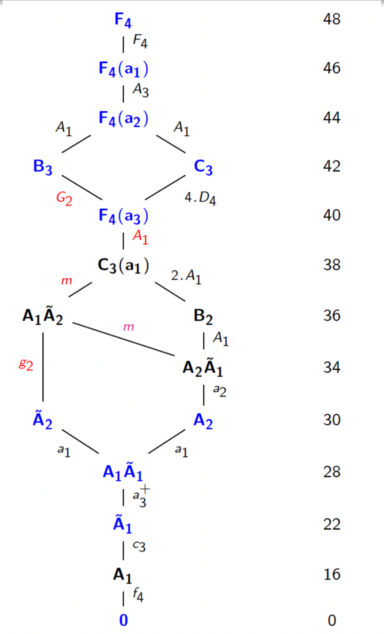

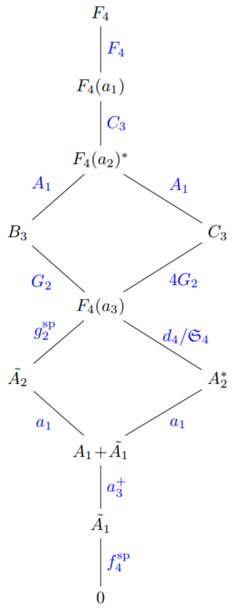

Langlands dual

- any reductive group G is associated to its Langlands dual ${}^L G$
- simply-connected goes to adjoint: $G^{sc} \leftrightarrow G^{ad}$ except:
- Sp_{2n} (simply-connected) is dual to SO_{2n+1} (adjoint)
- For simple Lie algebra $\mathfrak{g} = {}^L\mathfrak{g}$ except:
- $\mathfrak{so}_{2n+1}$ is dual to $\mathfrak{sp}_{2n}$.

G **and** ${}^L G$ **have the same Weyl group**.

Motivation from geometric Langlands

$$\begin{array}{ccc} \mathcal{M}_{SL_n} & & \mathcal{M}_{PSL_n} \\ \text{Hitchin} \downarrow & & \downarrow \\ \mathcal{A}_{SL_n} & \cong & \mathcal{A}_{PSL_n} \end{array}$$

[Hausel–Thaddeus] This is dual fibration!

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Conj [H-T] $E_{st}(\mathcal{M}_{SL_n}) = E_{st}(\mathcal{M}_{PSL_n})$

Proven by Gröchenig–Wyss–Ziegler

Maulik–Shen

Stringy E-functional

- Y: $\mathbb{Q}$-Gorenstein with klt singularities.
- $\pi : Z \to Y$ a log resolution with $\mathrm{Exc}(\pi) = \cup_{i \in I} D_i$.
- $\forall J \subset I$, set $D_J^\circ = \cap_{j \in J} D_j \setminus \cup_{i \notin J} D_i$.
- $E(D_J^\circ; u, v) = \sum_{p,q \geq 0} (-1)^{p+q} h_c^{p,q}(X) u^p v^q$.
- Write $K_Z = \pi^* K_Y + \sum_{j \in I} a_j D_j$.

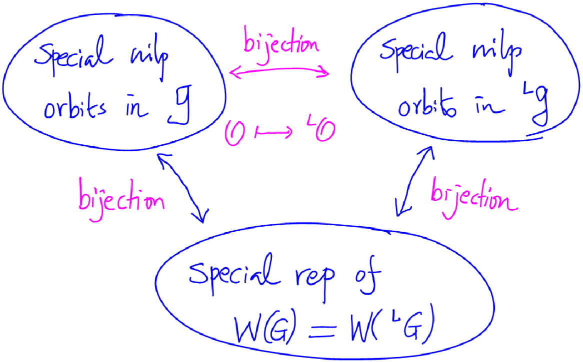
Definition

$$E_{st}(Y; u, v) = \sum_{J \subset I} E(D_J^\circ; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1}.$$

Proposition

- *independent of the choice of log resolution.*
- *if π is crepant, i.e. $K_Z = \pi^* K_Y$, then $E_{st}(Y; u, v) = E(Z; u, v)$.*

Springer dual of special orbits

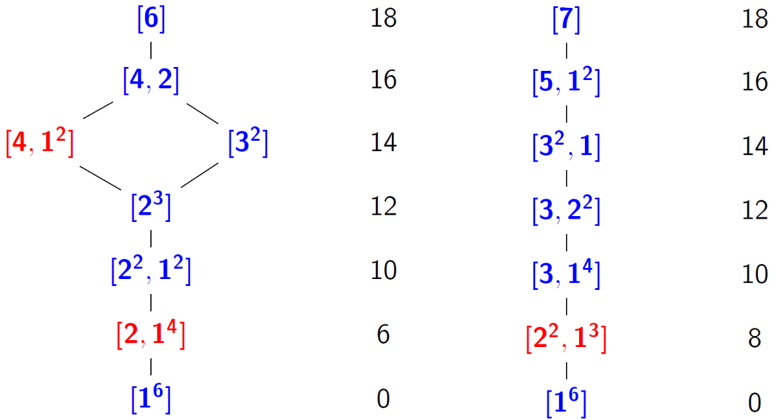


Spaltenstein: order-preserving and dim-preserving.

Problem (Motivated from geometric Langlands)

What can we expect about the geometry of $\mathcal{O}$ and its dual ${}^L\mathcal{O}$?

Nilpotent orbits in $\mathfrak{sp}_6$ and $\mathfrak{so}_7$



Special orbits in $\mathfrak{sp}_6$ and $\mathfrak{so}_7$

$$\begin{array}{ccccc} [6] & \to & [7] & & 18 \\ | & & | & & \\ [4,2] & \to & [5,1^2] & & 16 \\ | & & | & & \\ [3^2] & \to & [3^2,1] & & 14 \\ | & & | & & \\ [2^3] & \to & [3,2^2] & & 12 \\ | & & | & & \\ [2^2,1^2] & \to & [3,1^4] & & 10 \\ | & & | & & \\ [1^6] & \to & [1^6] & & 0 \end{array}$$

Naive guess

Naive Guess:

$$\begin{array}{ccc} Sp_{2n} & \xleftrightarrow[\text{dual}]{\text{Langlands}} & SO_{2n+1} \\ \cup & & \cup \\ \mathcal{O}_C & \xleftrightarrow[\text{dual}]{\text{Springer}} & \mathcal{O}_B \end{array}$$

then $E_{st}(\widetilde{\overline{\mathcal{O}}_C}) = E_{st}(\widetilde{\overline{\mathcal{O}}_B})$

Example

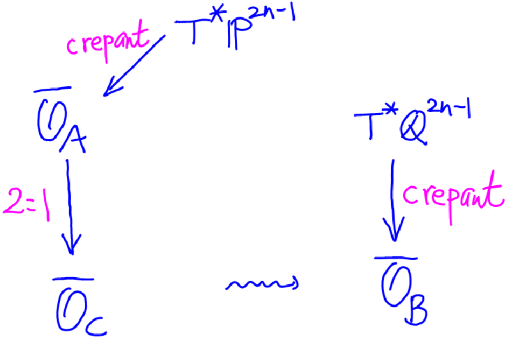
$$\mathcal{O}_C = \mathcal{O}_{[2^2 1^{2n-4}]} \rightsquigarrow \mathcal{O}_B = \mathcal{O}_{[3 1^{2n-2}]}$$

$E_{st}(\overline{\mathcal{O}}_B) = E(T^* Q^{2n-1})$ is polynomial $\Leftarrow$ $T^* Q^{2n-1} \downarrow$ crepant $\overline{\mathcal{O}}_B$

While $E_{st}(\overline{\mathcal{O}}_C)$ is not polynomial

So $E_{st}(\overline{\mathcal{O}}_B) \neq E_{st}(\overline{\mathcal{O}}_C)$

Rescue from double cover



While $E(T^*\mathbb{P}^{2n-1}) = E(T^*\mathbb{Q}^{2n-1})$

thus $E_{st}(\overline{\mathcal{O}}_A) = E_{st}(\overline{\mathcal{O}}_B)$

Slogan

Conj (Fu-Ruan-Wen)

$\exists$ equivariant finite covers

$$X \to \tilde{\mathcal{O}}_C \qquad {}^L X \to \tilde{\mathcal{O}}_B$$

s.t. $E_{st}(\tilde{\mathcal{O}}_C) = E_{st}({}^L X) \quad E_{st}(X) = E_{st}(\tilde{\mathcal{O}}_B)$

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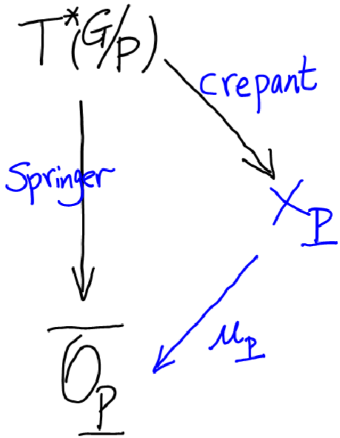
- **Main question**: which cover should appear?
- $\mathcal{O}$ and ${}^L\mathcal{O}$ have finite fundamental groups.

Conjecture holds for parabolic covers: Fu-Ruan-Wen



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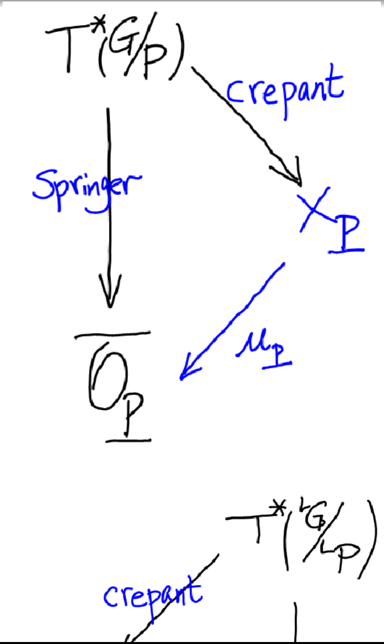

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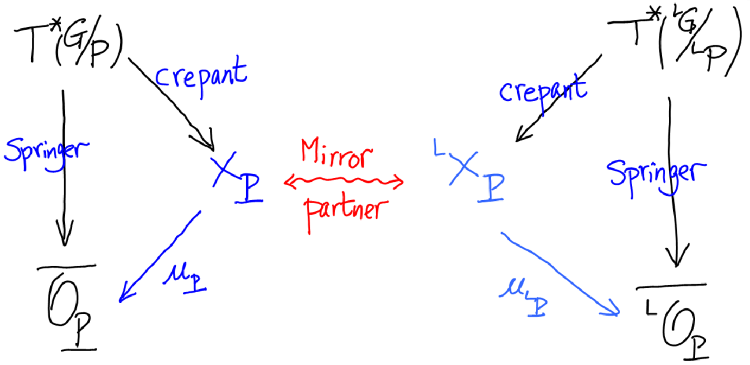

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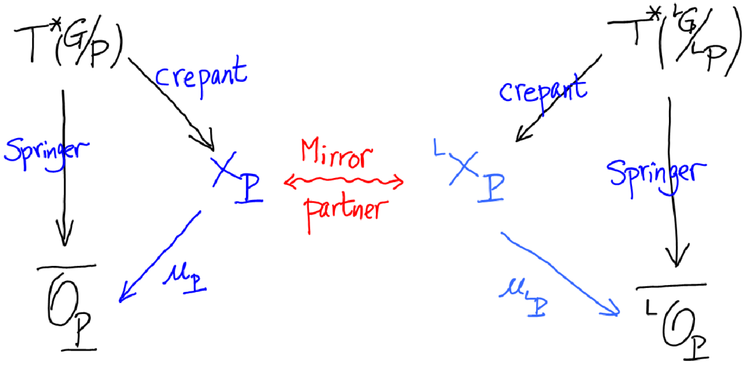


(i) $\deg \mu_P \cdot \deg \mu_{^LP} = |\bar{A}(\mathcal{O})|$

(seesaw property)

(ii) $E_{st}(X_P) = E_{st}(^LX_P)$

Special pieces

Definition

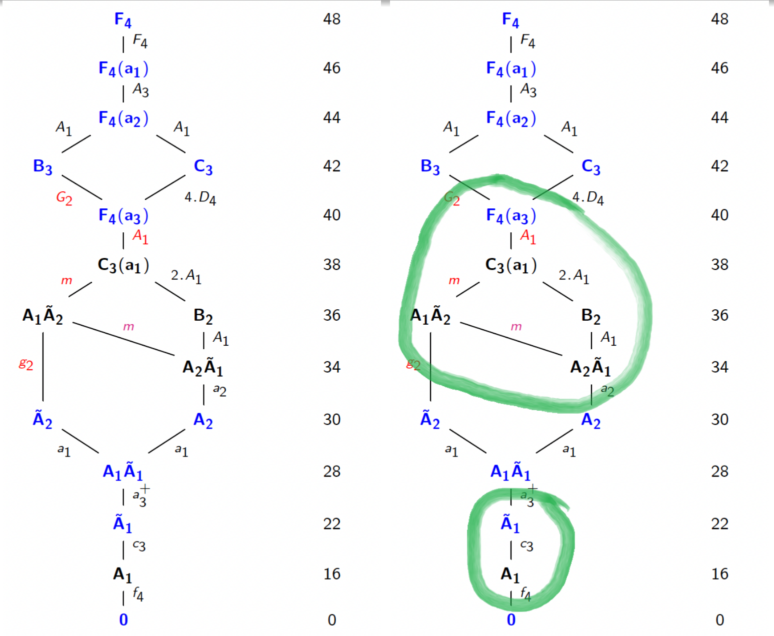
For a special nilpotent orbit $\mathcal{O}$, the special piece $\mathcal{P}(\mathcal{O})$ containing $\mathcal{O}$ is the locally-closed subvariety of $\mathfrak{g}$ given by:

$$\mathcal{P}(\mathcal{O}) := \overline{\mathcal{O}} - \bigcup_{\mathcal{O}' < \mathcal{O} \text{ special}} \overline{\mathcal{O}'}.$$

Theorem (Spaltenstein)

The special pieces give a partition of the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$.

Special orbits and pieces in F_4



Special pieces are rationally smooth

Conjecture (Lusztig 1981)

All special pieces $\mathcal{P}(\mathcal{O})$ are rationally smooth.

- trivial for type A.
- the minimal nonzero special piece (which covers G_2)
- Shoji '82: F_4 via the Green functions
- Beynon and Spaltenstein '84: all E_n cases
- Kraft and Procesi '89: for classical types, $\mathcal{P}(\mathcal{O})$ is a quotient of a smooth variety by a finite group.

Theorem (Lusztig '97)

For a special orbit $\mathcal{O} \subset \mathfrak{so}_{2n+1}$ *and its Langlands dual orbit* ${}^L\mathcal{O} \subset \mathfrak{sp}_{2n}$*, we have*

$$h_j^G(\mathcal{P}(\mathcal{O}), \mathbb{C}) = h_j^{{}^LG}(\mathcal{P}({}^L\mathcal{O}), \mathbb{C}), \quad \forall j.$$

Lusztig's special piece conjecture

Conjecture (Lusztig 1997)

Every special piece $\mathcal{P}(\mathcal{O})$ is a quotient Z/H of a smooth variety Z by a finite group H.

- Kraft and Procesi '89: OK for classical types
- Achar and Sage '09: proposed a candidate for Z by using the Deligne-Bezrukavnikov theory of perverse coherent sheaves, but smoothness is unknown.

Local geometry of special pieces

Theorem (F.-Juteau-Levy-Sommers)

A Slodowy transverse slice of a special piece is isomorphic to

- $\mathfrak{g}$ **classical**: $\prod_{i=1}^{r}(\mathbb{C}^{2k_i}/\pm 1)$.
- $\mathfrak{g}$ **exceptional**: $(\mathfrak{h}_{n-1} \oplus \mathfrak{h}_{n-1}^*)^k/\mathcal{S}_n$,

where $\mathfrak{h}_{n-1}$ is the reflection representation of $\mathcal{S}_n$. We have $k = 1$ always except one case in E_8 (where $k = 2$).

Corollary

Every special piece in the exceptional types is rationally smooth and normal.

Two orbits case: F.-Juteau-Levy-Sommers '17

If the special piece consists of two orbits, then it follows from:

Theorem (FJLS'17)

The generic singularity S of a nilpotent orbit in exceptional $\mathfrak{g}$ is

(a) *If* $\dim S = 2$, *then S is a union of surfaces of one of the following types*

- $A_k (k = 1, 2, 5), A_k^+ (k = 2, 4)$,
- $B_3, C_k (k = 2, 3, 4, 5, 6), D_6$,
- $E_k (k = 6, 7, 8), F_4, G_2$,
- m

(b) *If* $\dim S \geq 4$, *then it is equivalent to one of the following*

- $a_k (k = 1, 2, 5), a_k^+ (k = 2, 3, 4)$,
- $b_k (k = 2, 3, 4, 5, 6), c_k (k = 3, 4), d_6, d_4^{++}$,
- $e_k (k = 6, 7, 8), e_6^+$,

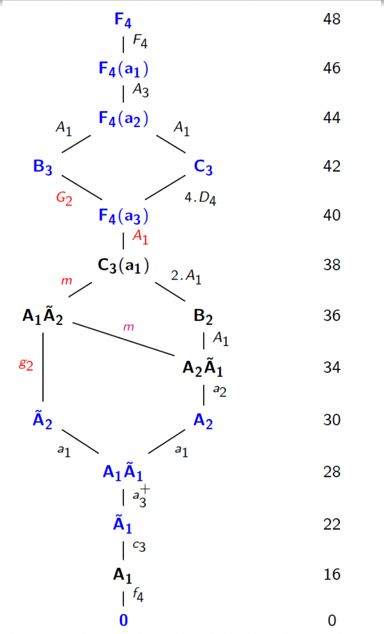
Typical special pieces with 3 orbits

$$\begin{array}{c} \mathbf{G_2(a_1)} \\ \mid A_1 \\ \tilde{\mathbf{A}}_1 \\ \mid m \\ \mathbf{A_1} \end{array} \qquad \qquad \begin{array}{c} \mathbf{D_4(a_1) + A_1} \\ \mid b_2 \\ \mathbf{A_3 + 2A_1} \\ \mid m' \\ \mathbf{2A_2 + 2A_1} \end{array}$$

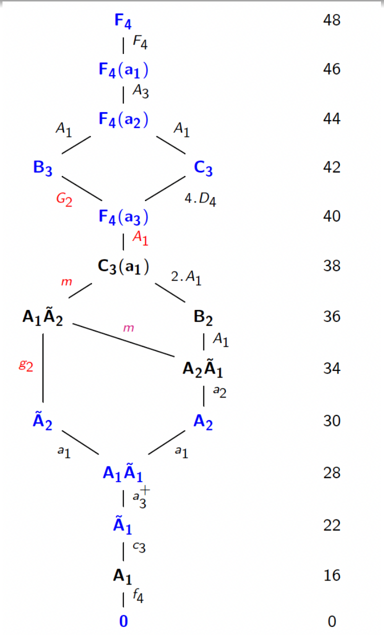
$$(\mathbb{C}^2 \oplus (\mathbb{C}^2)^*)/\mathcal{S}_3 \qquad \qquad (\mathbb{C}^2 \oplus (\mathbb{C}^2)^*)^2/\mathcal{S}_3$$

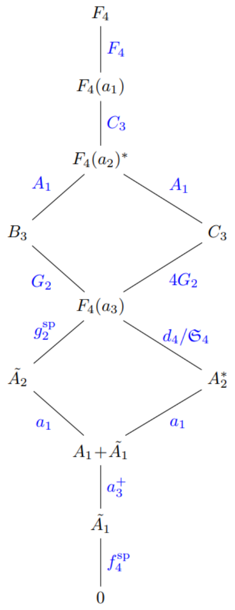
- m is pinched $\mathbb{C}^2$, i.e. $\mathrm{Spec}(\mathbb{C}[x,y]_{\geq 2})$
- m' is pinched $\mathbb{C}^4$, $\mathrm{Spec}(\mathbb{C}[x,y,s,t]_{\geq 2})$

Special orbits in F_4

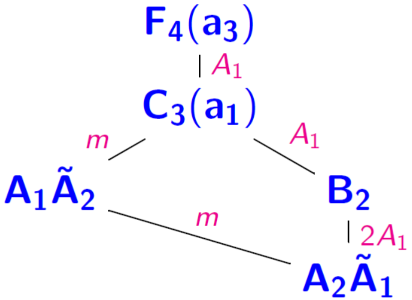


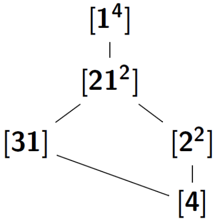
Special orbits in F_4



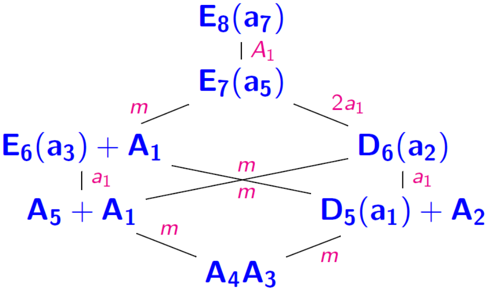


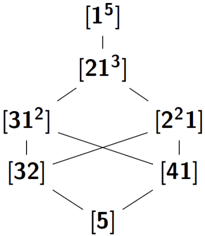
A special piece in $F_4(a_3)$





A special piece in E_8





$$(\mathbb{C}^4 \oplus (\mathbb{C}^4)^*)/\mathcal{S}_5$$


THANK YOU