# FUNCTIONAL TRANSCENDENCE ON BOUNDED SYMMETRIC DOMAINS

**Ngaiming Mok**

**The University of Hong Kong**

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# Moduli Space of Elliptic Curves

**An elliptic curve is complex-analytically a compact Riemann surface $S$ of genus 1.** In other words, $S := \mathbb{C}/L$ **for some lattice** $L \subset \mathbb{C}$. Replacing $L$ by $\lambda L$ for some $\lambda \in \mathbb{C} - \{0\}$, without loss of generality we may assume $L_\tau = \mathbb{Z} + \mathbb{Z}\tau$, $\mathrm{Im}(\tau) > 0$, i.e., $\tau \in \mathscr{H}$, where $\mathscr{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$, the upper half plane. Write $S_\tau := \mathbb{C}/L_\tau$.

For $\tau, \tau' \in \mathscr{H}$, we have $S_\tau \cong S_{\tau'}$ **if and only** if there exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$ such that $L_{\tau'} = \lambda L_\tau$, **i.e.**, if and only if $\tau' = \frac{a\tau+b}{c\tau+d}$ **where** $ad - bc \neq 0$. **Thus, the set of equivalence classes of $\mathbb{C}/L$ is in one-to-one correspondence with $X = X(1) := \mathscr{H}/\mathbb{P}SL(2,\mathbb{Z})$. We have the $j$-function $j : X(1) \xrightarrow{\cong} \mathbb{C}$, and $\overline{X(1)} = \mathbb{P}^1$.**

$\mathbb{P}SL(2,\mathbb{Z})$ acts on $\mathscr{H}$ with fixed points. One can introduce a refinement of the equivalence relation by enriching $\mathbb{C}/L$ with an additional piece of data.

# The $j$-function

**On the upper half plane H** $= \{\tau : \mathrm{Im}(\tau) > 0\}$ **define**

$$j(\tau) = 1728 \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^2} = 1728 \frac{g_2(\tau)^3}{\Delta(\tau)}$$

where $g_2(\tau) = 60 \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-4}$; $g_3(\tau) = 140 \sum_{(m,n)\neq(0,0)} (m+n\tau)^{-6}$.

and $\Delta(\tau) := g_2(\tau)^3 - 27 g_3(\tau)^2$ is the modular discriminant.

**The $j$-function establishes a biholomorphism $j : \mathbf{H}/\mathrm{SL}(2,\mathbb{Z}) \xrightarrow{\cong} \mathbb{C}$.**

Take $N > 1$ to be an integer. Then, the set of $N$-torsion points $\mathfrak{T}(n)$ of $S = \mathbb{C}/L_\tau$ is isomorphic to $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, but the isomorphism $\varphi : \mathfrak{T}(N) \xrightarrow{\cong} (\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z})^2$ is not unique. If we classify objects $(X, \varphi)$ consisting of an elliptic curve $X$ together with a choice of isomorphism of abelian groups $\varphi$, then we have more equivalence classes and the set of equivalence classes $X(N)$ of $(S, \varphi)$ is in one-to-one correspondence with $\mathscr{H}/\Gamma(N)$, where $\Gamma(N) \subset \mathbb{P}\mathrm{SL}(\mathbb{Z})$ consists of projectivizations of elements $T \in \mathrm{SL}(2, \mathbb{Z})$ such that $T \equiv I_2$ (the 2-by-2 identity matrix) modulo $N$.

**Write $X(N) := \mathbb{C}/\Gamma(N)$.** $X(n)$ classifies elliptic curves equipped with level $N$-structures. $\Gamma(N) \subset \Gamma(1) = \mathbb{S}L(2, \mathbb{Z})$ are called principal congruence subgroups. **For $N \geq 3$, $\Gamma(N)$ is torsion free, and $X(N)$ inherits from the Poincaré metric on the unit disk $\Delta$ a Hermitian metric of constant negative Gaussian curvature and finite volume.**

# Moduli Spaces of Polarized Abelian Varieties

Let $\tau, \tau' \in M_s(n, n, \mathbb{C})$ such that $\mathrm{Im}(\tau') > 0$ and $\mathrm{Im}(\tau) > 0$. Then $L_{\tau'} = \Lambda L_\tau$ for some $\Lambda \in \mathrm{GL}(n, \mathbb{C})$ if and only if there exists $\gamma \in \mathrm{Sp}(n, \mathbb{Z})$ such that $\tau' = \gamma\tau$. **Thus, the set of equivalence classes of PPAV is given by the quotient space $\mathcal{A}_n := \mathscr{H}_n / \mathbb{P}\mathrm{Sp}(n, \mathbb{Z})$.**

Let $N \geq 2$ be an integer. For an $n$-dimensional compact complex torus $\mathbb{T}$ **the abelian group of $N$-torsion points $\mathfrak{T}_N(\mathbb{T})$ is given by $\mathfrak{T}_N(\mathbb{T}) \cong (\mathbb{Z}/N\mathbb{Z})^{2n}$**. Consider the set of equivalence classes of $(A, \varphi)$ where $A$ is a principally polarized abelian variety and $\varphi : \mathfrak{T}_N(A) \xrightarrow{\cong} (\mathbb{Z}/N\mathbb{Z})^{2n}$. Denote by $\Gamma(N) \subset \mathrm{Sp}(n; \mathbb{Z})$ the subgroup of matrices $T$ congruent to $I_{2n}$ modulo $N$. **Then, the set $\mathcal{A}_n(N)$ of equivalence classes of $(A, \varphi)$ is given by $\mathscr{H}_n/\Gamma(N)$. For $N \geq 3$, $\Gamma(N) \subset \mathrm{Sp}(n, \mathbb{Z})$ is torsion-free.**

# Moduli Spaces of Polarized Abelian Varieties

$\mathcal{A}_n(N)$ is a complex manifold equipped with a quotient metric $g$ induced by the Bergman metric on the Siegel upper half plane $\mathscr{H}_n$. $(\mathcal{A}_n(N), g)$ **is a complete Kähler-Einstein manifold of nonpositive holomorphic bisectional curvature, negative Ricci curvature and finite volume**. By Satake compactification, $\mathcal{A}_n(g)$ **is canonically equipped with a quasi-projective structure obtained from Poincaré series.**

Imposing extra conditions on the PPAV $A_\tau = \mathbb{C}^n/L_\tau$, $\tau \in \mathscr{H}_n$, we obtain subvarieties of $\mathcal{A}_n(N)$ of the form $\mathscr{D}/\Gamma$ where $\mathscr{D} \subset \mathscr{H}_n$ is a totally geodesic complex submanifold biholomorphic to a bounded symmetric domain and $\Gamma \subset \mathrm{Aut}(\mathscr{D})$ is a lattice. **These are Shimura varieties. The prototype of such varieties were constructed by Shimura by requiring that** $A_\tau$ **has extra endomorphisms.** In general, $D/\Gamma$ **is called a Shimura variety whenever** $D \Subset \mathbb{C}^n$ **is a bounded symmetric domain and** $\Gamma \subset \mathrm{Aut}(D)$ **is an arithmetic lattice.**

# Bounded Symmetric Domains

**Classical cases**

$$D^I(p,q) = \{Z \in M(p,q,\mathbb{C}) : I - \overline{Z}^t Z > 0\}\ , \quad p,q \geq 1$$

$$D^{II}(n,n) = \{Z \in D^I_{n,n} : Z^t = -Z\}\ , \quad n \geq 2$$

$$D^{III}(n,n) = \{Z \in D^I_{n,n} : Z^t = Z\}\ , \quad n \geq 3$$

$$D^{IV}_n = \Big\{(z_1,\ldots,z_n) \in \mathbb{C}^n : \|z\|^2 < 2\ ;$$

$$\|z\|^2 < 1 + \Big|\tfrac{1}{2}\sum_{i=1}^n z_i^2\Big|^2\Big\}\ , \quad n \geq 3\ .$$

**Exceptional Domains**

$D^V$, $\dim 16$, type $E_6$

$D^{VI}$, $\dim 27$, type $E_7$

# Bergman Kernels for Classical Domains

$$K_{B^n}(z,w)=\frac{c_n}{(1-<z,w>)^{n+1}}\,;$$

$$K_{D^I(p,q)}(Z,W)=\frac{c_{p,q}}{\det\left(I_p-Z\overline{W}^t\right)^{p+q}}\,.$$

$$K_{D^{II}(n,n)}(Z,W)=\frac{a_n}{\det\left(I_n+Z\overline{W}\right)^{n-1}}\,.$$

$$K_{D^{III}(n,n)}(Z,W)=\frac{b_n}{\det\left(I_n-Z\overline{W}\right)^{n+1}}\,.$$

$$K_{D^{IV}_n}(z,w)=\frac{e_n}{\left(1-z\cdot\overline{w}+\frac{1}{4}\sum_{1\le i,j\le n}z_i^2\overline{w_j^2}\right)^n}$$

# Asymptotic Constancy of Holomorphic Sectional Curvatures on a Strictly Pseudoconvex Domain

**Theorem (Klembeck [Kl78])**

*Let $U \subset \mathbb{C}^n$ be a domain, $\rho$ be a smooth real function on $U$ and $b$ be a point on $U$. Suppose $\rho(b) = 0$ and $d\rho(x) \neq 0$ for any $x \in U$.* **Assume that $\rho$ is strictly plurisubharmonic on $U$, i.e., $\sqrt{-1}\partial\overline{\partial}\rho > 0$ on $U$.** *Let $U' \subset U$ be defined by $\rho < 0$, and $g$ be the Kähler metric on $U'$ whose Kähler form is given by $\omega = \sqrt{-1}\partial\overline{\partial}(-\log(-\rho))$.* **Then $(U', g)$ is asymptotically of constant holomorphic sectional curvature $-2$ at $b$.**

Klembeck's result is computational in nature. The following gives a suggestion why one should examine whether holomorphic sectional curvatures are asymptotically $-2$.

# Algebraic Extension of Holomorphic Isometries between Bounded Domains with Rational Bergman Kernels

**Theorem (Mok [Mo12, *JEMS*])**

*Let* $D \Subset \mathbb{C}^n$, resp. $\Omega \Subset \mathbb{C}^N$, *be bounded domains. Let* $x_0 \in D$, $\lambda \in \mathbb{R}, \lambda > 0$, *and* $f : (D, \lambda ds_D^2; x_0) \to (\Omega, ds_\Omega^2; f(x_0))$ *be a germ of holomorphic isometry. Suppose the Bergman kernel* $K_D(z, w)$ *extends as a rational function in* $(z, w)$ *and* $K_\Omega(\Xi, \xi)$ *extends as a rational function in* $(\Xi, \overline{\xi})$*. Then, the germ of* $\mathrm{Graph}(f) \subset D \times \Omega$ *at* $(x_0, f(x_0))$ *extends to an irreducible affine-algebraic subvariety* $S^\sharp \subset \mathbb{C}^n \times \mathbb{C}^N$*. If* $(\Omega, ds_\Omega^2)$ *is complete as a Kähler manifold, then* $S := S^\sharp \cap (D \times \Omega)$ *is the graph of a holomorphic isometric embedding* $F : (D, \lambda\, ds_D^2) \to (\Omega, ds_\Omega^2)$*. If furthermore* $(D, ds_D^2)$ *is complete, then* $F : D \to \Omega$ *is proper.*

# Asymptotic Total Geodesy of Embedded Poincaré Disks

**Theorem (Chan-Mok [CM21], *JDG*)**

*Let $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain in its Harish-Chandra realization equipping with the Bergman metric $ds^2_\Omega$. Let $\mu : U = \mathbb{B}^1(b_0, \epsilon) \to \mathbb{C}^N$ be a holomorphic embedding such that $\mu(U \cap \Delta) \subset \Omega$ and $\mu(U \cap \partial\Delta) \subset \partial\Omega$.* **Denote by $\sigma(z)$ the second fundamental form of $\mu(U \cap \Delta)$ in $(\Omega, ds^2_\Omega)$ at $z = \mu(w)$. Then, $\lim_{w \in U \cap \Delta,\, w \to b} \|\sigma(\mu(w))\| = 0$ for general point $b \in U \cap \partial\Delta$.**

**Theorem (Chan-Mok [CM21])**

*Let $f : (\Delta, \lambda ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ be a holomorphic isometric embedding, where $\lambda$ is a positive real constant and $\Omega \Subset \mathbb{C}^N$ is a bounded symmetric domain in its Harish-Chandra realization.* **Then, $f$ is asymptotically totally geodesic at a general point $b \in \partial\Delta$.**

# Why should Poincaré disks in BSDs be asymp. geodesic?

**Heuristics**: Argue by contradiction. Suppose otherwise. By rescaling one should be able to extract a nonstandard holomorphic isometry which is as close to being a equivariant holomorphic isometric embedding as possible.

**Conceptual Difficulty**: But a non-totally geodesic equivariant holomorphic isometric embedding of the Riemann Sphere with Gauss curvature $+1$ into $\mathbb{P}^n$ does exist, given by Veronese embeddings. To justify the heuristics one has to be able to make a distinction between model spaces of negative curvature (BSDs) against those of positive curvature.

**The Solution**: First we consider tube domains and embedded Poincaré disks where tangent vectors are of maximal rank. **We obtain a contradiction from the Poincaré-Lelong equation.** Then, we consider the general case by solving a **non-equidimensional integrability problem** of inserting a sufficiently small tube domain between **the Poincaré disk** and the **bounded symmetric domain**.

# Embedded Poincaré Disks with $\mathrm{Aut}(\Omega)$-equiv. Tangents

**Proposition** *Let* $f_0 : (\Delta, \lambda\, ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ *be a holomorphic isometric embedding. Suppose* $Z_0 := f_0(\Delta) \subset \Omega$ *is not asymptotically totally geodesic at a generic point* $b \in \partial Z_0$*.* **Then, there exists by rescaling a holomorphic isometric embedding** $f : (\Delta, \lambda\, ds^2_\Delta) \to (\Omega, ds^2_\Omega)$**,**
$f(\Delta) =: Z$ **with the following property.**
$(\dagger)$ **All tangent lines** $T_x(Z)$**,** $x \in Z$**, are equivalent under** $\mathrm{Aut}(\Omega)$**.**

**Proposition**

*Let $\Omega$ be an irreducible bounded symmetric domain of tube type and of rank $r$; $Z \subset \Omega$ be a local holomorphic curve with $\mathrm{Aut}(\Omega)$-equivalent tangent planes spanned by vectors of rank $r$.* **Then, $Z \subset \Omega$ is totally geodesic and of rank $r$ (i.e. of diagonal type).**

**Proof.** $\pi : \mathbb{P}T_\Omega \to \Omega$, $L \to \mathbb{P}T_\Omega$ tautological line bundle.
$[\mathscr{S}] \cong L^{-r} \otimes \pi^* E^2$, $E$ dual to $\mathcal{O}(1)$ on the compact dual $M$ of $\Omega$.

$$(2\pi)^{-1}\sqrt{-1}\partial\overline{\partial}\log\|s\|^2 = rc_1(L, \hat{g}_0) - 2c_1(\pi^*E, \pi^*h_0),$$

where $\hat{g}_0$ and $h_0$ are canonical metrics. $\|s(x)\|$ only depends on the $\mathrm{Aut}(\Omega)$-isomorphism type of $T_x(\Omega)$. Thus, $\|s\| =$ constant on $Z$. Hence,

$$0 = rc_1(L, \hat{g}_0) - 2c_1(\pi^*E, \pi^*h_0).$$

$\Leftrightarrow$ Gauss curvature $K(x) = -2/r$, and $\sigma \equiv 0$. $\square$

# Inserting a Total Geodesic Complex Submanifold $\Omega'$ so that Tangent Spaces of $Z \subset \Omega'$ are of Maximal Rank

## Proposition

*Let $\Omega$ be an irreducible bounded symmetric domain, $Z \subset \Omega$ be a local holomorphic curve with $\mathrm{Aut}(\Omega)$-equivalent tangent planes $T_x(\Omega) = \mathbb{C}\eta_x$.*
**Suppose rank$(\eta_x) =: k < r :=$ rank$(\Omega)$. Then, there exists a holomorphic vector bundle $W \subset T_\Omega\big|_Z := T$ such that**
**(a)** for the second fundamental form $\tau : T_Z \otimes W \to T/W$ we have $\overline{\partial}\tau\big|_{T_Z \times T_Z} \equiv 0$;
**(b)** as a consequence $W$ is parallel on $Z$;
**(c)** there exists a totally geodesic complex submanifold $\Omega' \subset \Omega$, such that $\Omega' \supset Z$ and $T_x(\Omega') = W_x$ for any $x \in Z$;
**(d)** $\Omega'$ is irreducible and rank$(\Omega') = k <$ rank$(\Omega)$.

# Inserting a Tube Domain Containing $Z$

**Proposition**

*Let $\Omega$ be an irreducible bounded symmetric domain, and $Z \subset \Omega$ be a local holomorphic curve with $\mathrm{Aut}(\Omega)$-equivalent tangent planes.* **Then, there exists a holomorphic vector subbundle $V \subset T_\Omega\big|_Z := T$ such that**

**(a)** For the second fundamental form $\tau : T_Z \otimes V \to T/V$ we have $\overline{\partial}\tau\big|_{T_Z \otimes T_Z} = 0$;

**(b)** as a consequence $V$ is parallel on $Z$;

**(c)** there exists a totally geodesic complex submanifold $\Omega' \subset \Omega$ such that $\Omega' \supset Z$ and $T_x(\Omega') = V_x$ for any $x \in Z$;

**(d)** $\Omega'$ is irreducible and of tube type.

# Application to Equivariant Holomorphic Isometries

As a first application we derive a rigidity result on equivariant holomorphic isometries, which was due to Clozel [Cl07] in the classical cases.

**Theorem (Chan-Mok [CM21])**

*Let $D$ and $\Omega$ be bounded symmetric domains, $\Phi : \mathrm{Aut}_0(D) \to \mathrm{Aut}_0(\Omega)$ be a group homomorphism, and $F : D \to \Omega$ be a $\Phi$-equivariant holomorphic map.* **Then, $F$ is totally geodesic.**

After reducing to the case where $D$ is irreducible, the latter theorem follows from the total geodesy of embedded Poincaré disks by polarization of the vanishing statements $\sigma(\gamma, \gamma) = 0$ of the second fundamental form $\sigma$ of $Z := F(D) \subset \Omega$ when restricted to **(a)** $\gamma = dF(\alpha)$ where $\alpha$ is tangent to a minimal disk $\Delta$ on $D$, **(b)** where $\gamma = dF(\beta)$ **where $\beta$ is a vector of rank 2 tangent to a degree-2 totally geodesic disk $\Delta'$**.

# A Result on Bi-Algebraicity

**Theorem (Ullmo-Yafaev [UY11])**

*Let $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain in its Harish-Chandra realization, $\Gamma \subset \mathrm{Aut}(\Omega)$ be an arithmetic torsion-free lattice. Write $X_\Gamma := \Omega/\Gamma$, $\pi : \Omega \to X_\Gamma$ for the uniformization map. Let $Y \subset X_\Gamma$ be an irreducible subvariety, and $Z \subset \Omega$ be an irreducible component of $\pi^{-1}(Y)$. Suppose $Z \subset \Omega$ is an algebraic subset.* **Then, $Z \subset \Omega$ is totally geodesic.**

The proof of [HY11] relies heavily on the monodromy result of André-Deligne [An92] on (quasi-)projective subvarieties of Shimura varieties, which in turn depends on Hodge structures and mixed Hodge structures. Such results are not known to be applicable in the case of nonarithmetic lattices.

# Algebraic Subsets of a Bounded Symmetric Domain Invariant under a Discrete Cocompact Group Action

**Theorem (Chan-Mok [CM21])**

*Let $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain in its Harish-Chandra realization, and $Z \subset \Omega$ be an algebraic subset. Suppose there exists a torsion-free discrete subgroup $\check{\Gamma} \subset \mathrm{Aut}(\Omega)$ such that $\check{\Gamma}$ stabilizes $Z$ and $Z/\check{\Gamma}$ is compact.* **Then, $Z \subset \Omega$ is totally geodesic.**

**Corollary (Chan-Mok [CM21])**

*Let $\Omega \Subset \mathbb{C}^N$ be as in Theorem, $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free cocompact lattice. Write $X_\Gamma := \Omega/\Gamma$, $\pi : \Omega \to X_\Gamma$ for the uniformization map. Let $Y \subset X_\Gamma$ be an irreducible subvariety, and $Z \subset \Omega$ be an irreducible component of $\pi^{-1}(Y)$. Suppose $Z \subset \Omega$ is an algebraic subset.* **Then, $Z \subset \Omega$ is totally geodesic.**

# Pseudo-homogeneity of $Z$ under a Complex Lie Group

For the Borel embedding of a bounded symmetric domain $\Omega \Subset \mathbb{C}^N \subset X_c$ into its compact dual we write $X_c = G/P$, where $G$ is the identity component of $\mathrm{Aut}(X_c)$, $G_0$ is the identity component of $\mathrm{Aut}(\Omega)$, $G_0 \subset G$ being a noncompact real form. We have

**Proposition**

*Let $Z \subset \Omega$ be an irreducible algebraic subset. Suppose there exists a torsion-free discrete subgroup $\check{\Gamma} \subset \mathrm{Aut}(Z)$ such that $\check{Y} = Z/\check{\Gamma}$ is quasi-projective. Let $H_0 \subset G_0$ be the identity component of the (positive-dimensional) stabilizer subgroup of $Z$, and $H \subset G$ be the complexification of $H_0$ inside $G$.* **Then, $Z$ is an irreducible component of $Hx \cap \Omega$ for any $z \in Z$.**

# Proof of Proposition by the Maximum Principle

**Proof.** There is a unique projective variety $\widehat{Z} \subset X_c$ such that $Z$ is an irreducible component of $\widehat{Z} \cap \Omega$. Since $G$ acts algebraically on $X_c$, the stabilizer subgroup $H_0 \subset G_0$ of $Z$ is algebraic. Since $\mathrm{Card}(\check{\Gamma}) = \infty$, $\dim(H_0) > 0$. Suppose Proposition fails. Then, there exists algebraic $E \subsetneq Z$ such that $E \supset Hx \cap Z$. There exists a projective subvariety $\widehat{E} \subset X_c$ such that $E$ is a finite union of irreducible components of $\widehat{E} \cap \Omega$. **Let now $P(z_1, \cdots, z_N)$ be a complex polynomial in $N$ variables ($N = \dim(\Omega)$) such that $P|_{\widehat{E} \cap \mathbb{C}^N} \equiv 0$ and such that $P|_{\widehat{Z} \cap \mathbb{C}^N} \not\equiv 0$.**

We derive a contradiction by the maximum principle. Define $\Phi : \Omega \to \mathbb{R}$ by $\Phi(z) = \sup\{|P(\gamma z)| : \gamma \in \check{\Gamma}\}$. Write $f_\gamma(z) := P(\gamma z)$ for $z \in \Omega$. Regarding $\{f_\gamma\}_{\gamma \in \check{\Gamma}}$ as a family of holomorphic functions on $\Omega$, We have the uniform bound $|f_\gamma(z)| \le \sup(|P||_{\overline{\Omega}}) < \infty$. **From Cauchy estimates, $\{f_\gamma\}_{\gamma \in \check{\Gamma}}$ is uniformly Lipschitz on any compact subset of $\Omega$, hence $\Phi$ is Lipschitz, in particular continuous, on $\Omega$. Thus, $\Phi$ is a bounded plurisubharmonic function on $\Omega$.** Restricting to $Z$ We have

By the definition of $\Phi$ we have $\Phi(\gamma z) = \Phi(z)$ for any $\gamma \in \check{\Gamma}$, hence we obtain by descent a nonconstant bounded plurisubharmonic function $\varphi$ on $\check{Y}$. Then, $\varphi$ is a nonconstant bounded plurisubharmonic function defined on $\check{Y}$. **By the Riemann extension theorem for bounded plurisubharmonic functions, $\varphi$ extends to a plurisubharmonic function, to be denoted by the same symbol, on smooth compactification $\overline{\check{Y}} \supset Y$. By the maximum principle for plurisubharmonic functions $\varphi$ must necessarily be a constant, a plain contradiction.**

Now, since the complex algebraic group $H \subset G$ acts algebraically on $X_c$, the Zariski closure of $Hx$ in $\widehat{Z} \subset X_c$ is the same as its topological closure, and we conclude that $\overline{Hx} \cap Z = Z$. **Suppose now $Hx \cap Z \subsetneq Z$ and let $y \in Z - Hx$. The same argument applies to $y$ (in place of $x$) and we have $\overline{Hy} \cap Z = Z$, contradicting with the fact that $Hx$ and $Hy$ are distinct and hence disjoint orbits.** We conclude that $Hx \cap Z = Z$ for any $x \in Z$. Hence, $Z$ is an irreducible component of $Hx \cap \Omega$, as desired. $\square$

# Nadel's Semisimplicity Theorem

## Theorem (Nadel [Na90])

*Let $X$ be a compact Kähler manifold with ample canonical line bundle, and denote by $\pi : \widetilde{X} \to X$ the uniformization map.* **Then, $\mathbf{Aut}_0(\widetilde{X})$ is a semisimple Lie group without compact factors.**

**Lemma** *Let $Z \subset \Omega$ be an algebraic subset, and let $\Omega' \subset \Omega$ be the smallest totally geodesic complex submanifold containing $Z$. Suppose $\gamma \in \mathrm{Aut}(\Omega')$ such that $\gamma|_Z = \mathrm{id}_Z$. Then, $\gamma = \mathrm{id}_{\Omega'}$.*

## Proposition

*Suppose there exists a torsion-free discrete subgroup $\check{\Gamma} \subset \mathrm{Aut}(\Omega)$ such that $\check{\Gamma}$ stabilizes $Z$ and $Z/\check{\Gamma}$ is compact. Let $H_0 \subset \mathrm{Aut}(\Omega)$ be the identity component of the subgroup of $\mathrm{Aut}(\Omega)$ which stabilizes $Z$.* **Then, $H_0 \subset \mathbf{Aut}(\Omega)$ is a semisimple Lie group without compact factors.**

# Bi-algebraicity by means of Nadel's Theorem

## Maps inducing the representation $\theta : \check{\Gamma} \hookrightarrow H_0 \subset G_0 = \mathrm{Aut}_0(\Omega)$

**Without loss of generality assume $(i_Y)_*\pi_1(Y) = \check{\Gamma} \subset H_0$, $i_Y : Y \hookrightarrow Z_{\check{\Gamma}}$. By Nadel's Theorem, $H_0$ is a semisimple Lie group without compact factors which acts on $\Omega$. The homomorphism $\theta : \check{\Gamma} \hookrightarrow H_0 \subset G_0$ is a discrete representation.** Write $L \subset H_0$ for a maximal compact subgroup. Let $f : Y \to \check{\Gamma}\backslash H_0/L =: S_{\check{\Gamma}}$ be any smooth map which induces the representation $\theta$. **Since $(\Omega, ds^2_\Omega)$ is a Cartan-Hadamard manifold, i.e., a simply connected complete Riemannian manifold of nonpositive Riemannian sectional curvature, the center of gravity argument gives a point $x \in \Omega$ which is fixed by $L$.**

Regard $H_0/L$ as the orbit $H_0x \subset \Omega = G_0/K$, $L \subset K = \mathrm{Isot}_x(\Omega, ds^2_\Omega)$, hence $S_{\check{\Gamma}} \hookrightarrow X_{\check{\Gamma}} := \check{\Gamma}\backslash\Omega = \check{\Gamma}\backslash G/K$ as a real analytic submanifold.

# Total Geodesy of $Z \subset \Omega$

**Since $X_{\check{\Gamma}}$ is a $K(\pi, 1)$, the two smooth maps $f, g : Y \to X_{\check{\Gamma}}$ inducing the representation $\theta$ are homotopic to each other.**

Denote by $\omega$ the Kähler form of the canonical KE metric on $X_{\check{\Gamma}}$. $H_0$ acts on $\Omega$. For any $x \in X$, we have

$$\dim_{\mathbb{R}}(S_{\check{\Gamma}}) \leq \dim_{\mathbb{R}}(H_0 x) \leq \dim_{\mathbb{R}} Z = \dim_{\mathbb{R}} Y := 2m.$$

By homotopy $\int_Y (g^*\omega)^m = \int_Y (f^*\omega)^m$. The first integral gives $m!\mathrm{Vol}(Y, \omega|_Y) > 0$. **A contradiction would arise if we had strict inequality of dimensions.** Hence, equality holds, $Z$ is homogeneous under $H_0$, and $H_0$ is of Hermitian type. **Thus, $Z \subset \Omega$ is the image of an equivariant holomorphic map between bounded symmetric domains. By [CM21], $Z \subset \Omega$ is totally geodesic.**

# The Hyperbolic Ax-Lindemann Theorem

The following result is the solution to **the Hyperbolic Ax-Lindemann Conjecture**, the geometric component of **the André-Oort Conjecture on Shimura varieties** according to the strategy of Pila-Zannier. It was preceded by the work of Ullmo-Yafaev [UY14] in the case of cocompact lattices, and the work of Pila-Tsimerman [PT14] in the case of Siegel modular varieties.

**Theorem (Klingler-Ullmo-Yafaev [KUY16])**

*Let $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain in its Harish-Chandra realization, $\Gamma \subset \mathrm{Aut}(\Omega)$ be an arithmetic torsion-free lattice. Write $X_\Gamma := \Omega/\Gamma$, $\pi : \Omega \to X_\Gamma$ for the uniformization map. Let $Z \subset \Omega$ be an irreducible algebraic subset and denote by $\mathscr{Z} = \overline{\pi(Z)}^{\mathscr{Z}ar} \subset X_\Gamma$ the Zariski closure of image of $Z$ under the uniformization map in the quasi-projective variety $X_\Gamma$.* **Then, $\mathscr{Z} \subset X_\Gamma$ is a totally geodesic subset.**

# The Hyperbolic Ax-Lindemann Conjecture for General Lattices

**Theorem (Mok [Mo19, *Compositio Math.*])**

*Let $n \geq 2$ and $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ be a not necessarily arithmetic torsion-free lattice. Write $X_\Gamma := \mathbb{B}^n/\Gamma$, $\pi : \Omega \to X_\Gamma$ for the uniformization map. Let $Z \subset \Omega$ be an irreducible algebraic subset and denote by $\mathscr{Z} = \overline{\pi(Z)}^{\mathscr{Z}ar} \subset X_\Gamma$ be the Zariski closure of image of $Z$ under the uniformization map in the quasi-projective variety $X_\Gamma$.* **Then, $\mathscr{Z} \subset X_\Gamma$ is a totally geodesic subset.**

The analogue of the hyperbolic Ax-Lindemann conjecture for general (torsion-free) lattices $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ is expected to hold. The above result of [Mo17] gives a strong piece of evidence. The proof is differential-geometric in nature.

# The Hyperbolic Ax-Lindemann Theorem for Rank-1 Lattices: Scheme of Proof for Cocompact Lattices

**(a)** Consider the Borel embedding $\mathbb{B}^n \subset \mathbb{P}^n$, a Chow Component $\mathcal{K}$ on $\mathbb{P}^n$ containing $[Z]$ as a member, and the associated universal family $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to \mathbb{P}^n$. **The algebraic group $\mathrm{Aut}(\mathbb{P}^n)$ acts on $\mathcal{K}$; $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ acts on $\mathcal{U}_0 = \mathcal{U}|_{\mathbb{B}^n}$ and $\mathcal{U}_0$ descends to $\mu_\Gamma : \mathcal{U}_\Gamma \to X_\Gamma$.**

**(b)** $\mathcal{U}_\Gamma$ admits a tautological meromorphic foliation whose local leaves arise from tautological lifings of $\mu : \mathcal{U} \to \mathbb{P}^n$ via the covering maps $\pi_\Gamma : \mathbb{B}^n \to X_\Gamma$ and $\varpi_\Gamma : \mathcal{U}_0 \to \mathcal{U}_\Gamma$.

**(c)** Let $\widetilde{\mathscr{Z}}$ be an irreducible component of $\pi_\Gamma^{-1}(\mathscr{Z})$. Then, at a good point $b \in \partial\widetilde{\mathscr{Z}}$, $\widetilde{\mathscr{Z}}$ **extends across $b$ as the union of an analytic family of algebraic subvarieties of $\mathbb{P}^n$**. Let $\mathscr{D}$ be a germ of complex submanifold at $b$ grafted to extend $\widetilde{\mathscr{Z}}$ analytically across $b$.

# Scheme of Proof for Cocompact Lattices

**(d)** Choose canonical Kähler-Einstein metrics on $\mathbb{B}^n$ so that minimal disks are of constant Gaussian curvature $-2$. $\mathscr{D} \cap \mathbb{B}^n$ is a local strictly peudoconvex manifold with smooth boundary, and **by Klembeck [Kl87] $\mathscr{D} \cap \mathbb{B}^n$ is asymptotically of constant holomorphic sectional curvature $-2$, hence asymptotically totally geodesic.**

**(e)** Let $\check{\Gamma} \subset \Gamma$ be the subgroup of the image of $\pi_1(\mathscr{Z})$ in $\pi_1(X_\Gamma) = \Gamma$ consisting of elements which stabilize $\widetilde{\mathscr{Z}}$. Let $\Pi \subset \widetilde{\mathscr{Z}}$ be a compact fundamental domain with respect to $\check{\Gamma}$. By rescaling via elements $\gamma$ belonging to $\check{\Gamma}$, it follows that **$\Pi$ is of constant holomorphic sectional curvature $-2$, hence $\widetilde{Z} \subset \mathbb{B}^n$ is a totally geodesic complex submanifold.**

# Additional Ingredients for Non-cocompact Lattices

## Compactification of $X_\Gamma$

By Satake-Baily-Borel, in the arithmetic case $X_\Gamma = \Omega_\Gamma$ always admits a projective compactificaton. In the arithmetic rank-1 case, the Satake compactification is obtained by adjoining a finite number of normal isolated singularities. Siu-Yau (1982) showed that a complete Kähler manifold $X$ of finite volume and pinched negative holomorphic bisectional curvature always admits a (Moishezon) compactification by adding a finite number of normal isolated singularities and Mok (2012) showed that in the case where $X = X_\Gamma$ is a rank-1 finite-volume quotient manifold, the compactification thus obtained is projective.

**Theorem (Mok-Zhong [MZ89, *Ann. Math.*])** (Compactification Theorem by $L^2$-estimates of $\overline{\partial}$)

*Let* $(X, g)$ *be a complete Kähler manifold.* **Assume that** $\mathrm{Vol}(X, g) < \infty$, $\|$**Sectional Curvature**$(X, g)\| < \infty$ *, and that* $X$ *is homotopic to a finite CW-complex.* **Suppose there exists a Hermitian holomorphic line bundle** $(E, h)$ **of pinched positive curvature.** *For* $k > 0$ *, denote by* $\mathcal{N}(X, E^k)$ *the space of holomorphic sections* $s \in \Gamma(X, E^k)$ *of* **the Nevanlinna class**, *i.e., s satisfies* $\int_X \max(\log \|s\|_{h^k}, 0) < \infty$. *Then, for each positive integer* $k$, *we have* $\dim(\mathcal{N}(X, E^k)) < \infty$. **Moreover, there exists some positive integer** $k$ **such that** $\mathcal{N}(X, E^k)$ **has no base points and it embeds** $X$ **into** $\mathbb{P}(\mathcal{N}(X, E^k)^*)$ **realizing** $X$ **as a quasi-projective manifold.**

## Natural holomorphic fiber bundles over $X_\Gamma$

Write $G_0 = \mathrm{Aut}(\mathbb{B}^n)$, $\mathcal{U}_0 = \mathcal{U}|_{\mathbb{B}^n}$, etc. The total space of $\mu_0 : \mathcal{U}_0 \to \mathbb{B}^n$ is a $G_0$-equivariant holomorphic fiber bundle which is naturally embedded in a $G_0$-equivariant holomorphic projective bundle. $\theta_0 : \mathscr{P}_0 \to \mathbb{B}^n$ is thus equipped with a $G_0$-invariant Kähler metric $\theta_0$ which descends to a Kähler metric $\theta_\Gamma$ on the total space of $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$.

## Compactification of the projective bundle $\mathscr{P}_\Gamma$

$(\mathscr{P}_\Gamma, \theta_\Gamma)$ is a complete Kähler manifold, $\mathrm{Vol}(\mathscr{P}_\Gamma, \theta_\Gamma) < \infty$. By local homogeneity, $(\mathscr{P}_\Gamma, \theta_\Gamma)$ is of bounded sectional curvature, and $(P_\Gamma, \theta_\Gamma)$ admits a locally homogeneous Hermitian holomorphic line bundle $(L_\Gamma, h_\Gamma)$ of positive curvature. Moreover, $X_\Gamma$ and hence $P_\Gamma$ is of finite topological type. Hence, by $L^2$-estimates of $\overline{\partial}$, $(\mathscr{P}_\Gamma, \theta_\Gamma)$ admits a projective compactification realizing $\mathscr{P}_\Gamma$ as a quasi-projective manifold.

## Compactification of $\mathcal{U}_\Gamma$

For each $x \in X_\Gamma$, the fiber $\mathcal{U}_{\Gamma,x} \subset \mathscr{P}_{\Gamma,x}$ corresponds to an element $[U_{\Gamma,x}] \in \mathrm{Chow}(\mathscr{P}_{\Gamma,x}) \subset \mathrm{Chow}(\overline{\mathscr{P}_\Gamma})$, where $\overline{\mathscr{P}_\Gamma} \supset \mathscr{P}_\Gamma$ denotes a projective compactification. Denote by $\mathscr{S}$ the irreducible component of $\mathrm{Chow}(\overline{\mathscr{P}_\Gamma})$ containing the points $[U_{\Gamma,x}]$ as $x$ ranges over $X_\Gamma$. **Then $\mathscr{S}$ is projective, and the assignment $\varphi(x) = [U_{\Gamma,x}] \in \mathscr{S} \subset \mathbb{P}^N$ (for some $N$) defines a holomorphic map into a projective space.** Given that $n \geq 2$, and the minimal compactification $\overline{X_\Gamma}_{\mathrm{min}}$ is a normal projective variety obtained by adding a finite number of isolated singularities, by **Hartogs Extension $\varphi$ extends meromorphically to $\overline{X_\Gamma}_{\mathrm{min}}$**. Hence, the total space of $\mu_\Gamma : \mathcal{U}_\Gamma \to X_\Gamma$ extends as a projective subvariety $\overline{\mathcal{U}_\Gamma} \subset \overline{\mathscr{P}_\Gamma} \subset \mathscr{S}$.

1. The use of Hartogs Extension simplifies the argument. Compactification of $\mathcal{U}_\Gamma$ works in general by means of [MZ89].
2. For the foliation argument one may need to work with a substratum $\nu_\Gamma : \mathcal{V}_\Gamma \to X_\Gamma$ of $\mathcal{U}_\Gamma$. The compactification argument is unchanged.

# Hyperbolic Ax-Schanuel for Arithmetic Lattices

**Theorem (Mok-Pila-Tsimerman ([MPT19, *Annals*]))**

*Let $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain, $\Gamma \subset \mathrm{Aut}(\Omega)$ be an arithmetic lattice, and write $X_\Gamma := \Omega/\Gamma$, as a quasi-projective variety. Let $W \subset \Omega \times X_\Gamma$ be an algebraic subvariety. Let $D \subset \Omega \times X_\Gamma$ be the graph of the uniformization map $\pi_\Gamma : \Omega \to X_\Gamma$, and* $U$ **be an irreducible component of** $W \cap D$ **whose dimension is larger than expected**, *i.e.,*

$$\mathrm{codim}(U) < \mathrm{codim}(W) + \mathrm{codim}(D),$$

*the codimensions being in $\Omega \times X_\Gamma$, or, equivalently,*

$$\dim(U) > \dim(W) - \dim(X_\Gamma).$$

**Then, the projection of $U$ to $X_\Gamma$ is contained in a totally geodesic subvariety $Y \subsetneq X_\Gamma$.**

# Hyperbolix Ax-Schanuel $\Rightarrow$ Hyperbolic Ax-Lindemann

Let $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free arithmetic lattice, $\pi : \Omega \to \Omega/\Gamma =: X_\Gamma$ be the universal covering map. Let $Z \subset \Omega$ be an irreducible algebraic subset, and $Y \subset X_\Gamma$ be the Zariski closure of $\pi(Z)$. **The Hyperbolic Ax-Lindemann Conjecture asserts that $Y$ is totally geodesic in $X_\Gamma$.**

**To *deduce* Hyperbolic Ax-Lindemann, consider $Z \subset \Omega$ such that $Y \subsetneq X_\Gamma$.** Write $W := Z \times Y \subset \Omega \times X_\Gamma$. Then. $W$ is algebraic. Recall that $D = \mathrm{Graph}(\pi)$. Let $\widetilde{Y}$ be an irreducible component of $\pi^{-1}(Y)$ containing $Z$. Then, $W = Z \times Y \subset \widetilde{Y} \times Y$. Write $D_{\widetilde{Y}} := \mathrm{Graph}(\pi|_{\widetilde{Y}}) \subset \widetilde{Y} \times Y$. Thus, $W \cap D = W \cap D_{\widetilde{Y}}$, and

$$\dim(W \cap D) = \dim(W \cap D_{\widetilde{Y}}) \geq \dim(W) + \dim(D_{\widetilde{Y}}) - \dim(\widetilde{Y} \times Y)$$
$$= \dim(W) + \dim(Y) - 2\dim(Y) = \dim(W) - \dim(Y)$$

# Hyperbolic Ax-Schanuel $\Rightarrow$ Hyperbolic Ax-Lindemann

If we let $U$ be any irreducible component of $W \cap D = W \cap D_{\widetilde{Y}}$, then the same argument applies to $U$, and we have

$$\dim(U) \geq \dim(W) - \dim(Y) > \dim(W) - \dim(X_\Gamma),$$

hence by hyperbolic Ax-Schanuel $\pi(U)$ is contained in some totally geodesic subset $\Pi_0 \subsetneq X_\Gamma$. Let now $\Pi \subset X_\Gamma$ be the intersection of all totally geodesic subset $\Pi_0 \subset X_\Gamma$ such that $Y \subset \Pi_0 \subset X_\Gamma$. Write $\Pi = \pi(\Omega')$, where $\Omega' \subset \Omega$ is a totally geodesic complex submanifold, hence a bounded symmetric domain. $\Pi \subsetneq X_\Gamma$ is the smallest totally geodesic subset containing $Y$. Applying the same argument to $\Omega'$ in place of $\Omega$ again by the hyperbolic Ax-Schanuel, if $Y \subsetneq \Pi$ then there is a totally geodesic subset $\Pi' \subsetneq \Pi$ such that $Y \subset \Pi'$, contradicting minimality of $\Pi$. We conclude that actually $Y = \Pi \subset X_\Gamma$, proving hyperbolic Ax-Lindemann.

# Ax-Schanuel in terms of functional transcendence

Fix a torsion-free lattice $\Gamma \subset \mathrm{Aut}(\Omega), \pi : \Omega \to X_\Gamma$. Modular functions are $\Gamma$-invariant meromorphic functions descending to rational functions on $X_\Gamma$.

**Theorem (Mok-Pila-Tsimerman ([MPT19, *Annals*]))**

Let $V \subset \Omega$ be an irreducible complex analytic variety, not contained in any weakly special subvariety $E \subsetneq \Omega$. Let $\{z_i, i = 1, \dots, n\}$ be algebraic coordinates on $\Omega$. Let $\{\phi_1, \dots, \phi_N\}$ be a basis of modular functions. Then,

$$\mathrm{trans.deg.}_{\mathbb{C}}\mathbb{C}\big(\{z_i\}, \{\phi_j\}\big) \geq n + \dim V.$$

where all $\phi_j$ are assumed defined at some point on $V$ and restricted to $V$.

1. We may take the algebraic coordinates $\{z_i, i = 1, \dots, n\}$ to be the Harish-Chandra coordinates on $\Omega \Subset \mathbb{C}^n \subset \widehat{\Omega}$.
2. Here a weakly special subvariety $E \subset \Omega$ is a totally geodesic submanifold $E \subset \Omega$ such that $\pi(E) \subset X_\Gamma$ is quasi-projective.