Correction:

Theorem (Witten, Taubes) Let X be a symplectic 4-mfd with $b^+(X)>1$. Then X has no psc metric.

$(b^+(X)>0$ ~~X~~ $)$

$\mathbb{CP}^2$

$b^+ = \#$ positive e.v. of Q_X

Lecture 5. A crashed course on Morse theory

X: closed manifold, smooth $\quad f: X \to \mathbb{R}$ smooth

$p \in X$ is called a "critical point" if $df_p: T_pX \to T_{f(p)}\mathbb{R} = \mathbb{R}$ is trivial.

$\mathrm{crit}(f) = \{\text{critical points of } f\}$

$r \in \mathbb{R}$ is called a critical value if $r = f(p)$ for some $p \in \mathrm{Crit}(f)$.

otherwise, we call r a "regular value"

For $p \in \mathrm{Crit}(f)$, $\exists$ well-defined $\mathrm{Hess} f_p: T_pX \otimes T_pX \to \mathbb{R}$

defined by $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$.

We say $p \in \mathrm{Crit}(f)$ is $\boxed{\text{non-degenerate}}$ if $\det(\mathrm{Hess} f_p) \neq 0$.

We define $\mathrm{index}(p) := \#$ negative eigenvalues of $\mathrm{Hess} f_p$.

$\mathrm{Crit}_k(f) := \{\text{index-}k \text{ critical points}\}$

Definition: We say f is a $\boxed{\text{Morse function}}$ if all critical points are non-degenerate. If $\partial X \neq \phi$, we require $f^{-1}(\max f) = \partial X$.

Fact: Any smooth function can be perturbed into a Morse function.

Morse lemma: Near a non-degenerate critical point, we can find local chart on X s.t under this local chart, we have

$$f(x_1, \cdots, x_n) = -x_1^2 - \cdots - x_{\mathrm{index}(p)}^2 + \cdots + x_n^2$$

$$-x_1^2 + x_2^2 + \cdots + x_n^2$$

We denote $f^{-1}(-\infty, r]$ by $X_{\le r}$.

When r is a regular value, $f^{-1}(-\infty, r]$ is a manifold with boundary $f^{-1}(r)$.

Exercise: Suppose $[a,b]$ contains no critical values. Then $X_{\le a} \cong_{diff} X_{\le b}$.

Q: What happens to $X_{\le r}$ when r crosses a critical value?

$$X_{\le r} = f^{-1}(-\infty, r]$$

$\boxed{\text{Handle attachment}}$

- Y: n-dim manifold with boundary
- $\varphi : S^{k-1} \to \partial Y$ smooth embedding
- A framing of φ: a trivilization of the normal bundle $N_{\varphi(S^{k-1})}^{\partial Y}$.
- Using the framing, we can define

$$\tilde{\varphi} : S^{k-1} \times D^{n-k} \xrightarrow{\cong_{diff}} \underbrace{\nu(\varphi(S^k))}_{\text{tubular neighborhood}}$$

- We say Y' is obtained from Y by attaching a $\overset{n\text{-dim}}{k}$-handle along $\varphi(S^{k-1})$ if $Y' = Y \cup_{\tilde{\varphi}} (D^k \times D^{n-k})$. Here

$$\begin{array}{ccc} & S^{k-1}\times D^{n-k} & \\ \swarrow^{\tilde{\varphi}} & & \searrow \\ \nu(\varphi(S^{k-1})) & & D^k \times D^{n-k} \end{array}$$

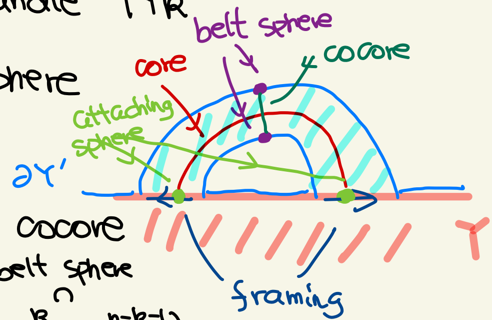
$$\partial(D^k \times D^{n-k}) = (S^{k-1} \times D^{n-k}) \cup (\underset{\cup}{\overset{0 \times S^{n-k-1}}{D^k \times S^{n-k-1}}})$$

$D^k \times D^{n-k}$: the (n-dimensional) k-handle H_k

$S^{k-1} \times 0 = \varphi(S^{k-1})$: the attaching sphere

$0 \times S^{n-k-1}$: the belt sphere

$D^k \times 0$: core $\qquad 0 \times D^{n-k}$: cocore



Note $\partial Y' = (\underbrace{\partial Y \setminus (S^{k-1} \times \mathring{D}^{n-k})}_{\supset\ \text{attachig sphere}}) \cup (\underbrace{D^k \times S^{n-k-1}}_{\supset\ \text{belt sphere}})$

So $\partial Y \xrightarrow{\text{surgery along the attachig sphere}} \partial Y'$

We use S_A^{k-1} to denote attachig sphere for H_k $\quad S_A^{k-1} \hookrightarrow \partial Y$

S_B^{n-k-1} to dnote belt sphere for H_k $\quad S_B^{n-k-1} \hookrightarrow \partial Y'$

We can attach multiple handles $H_{k_1}, \dots, H_{k_\ell}$ to Y at the same time:

$$Y' = Y \bigcup_{\substack{\sqcup \tilde{\varphi}_i \\ 1 \le i \le \ell}} (H_{k_1} \sqcup \cdots \sqcup H_{k_\ell})$$

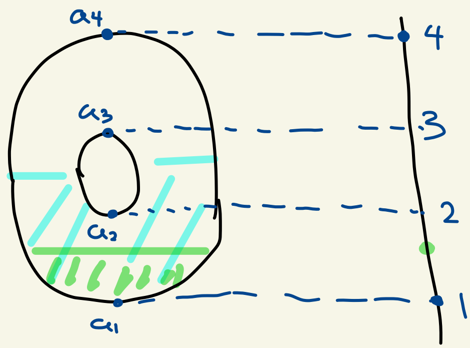
The attaching spheres $\{S_A^{k_i - 1}\}$ must be disjoint.

$\boxed{\text{Proposition:}}$ Suppose $f^{-1}(r) \cap \operatorname{crit}(f) = \{x_1, \dots, x_\ell\}$. Let $a_i := \operatorname{index}(x_i)$. Then for $\varepsilon > 0$ small enough, we have

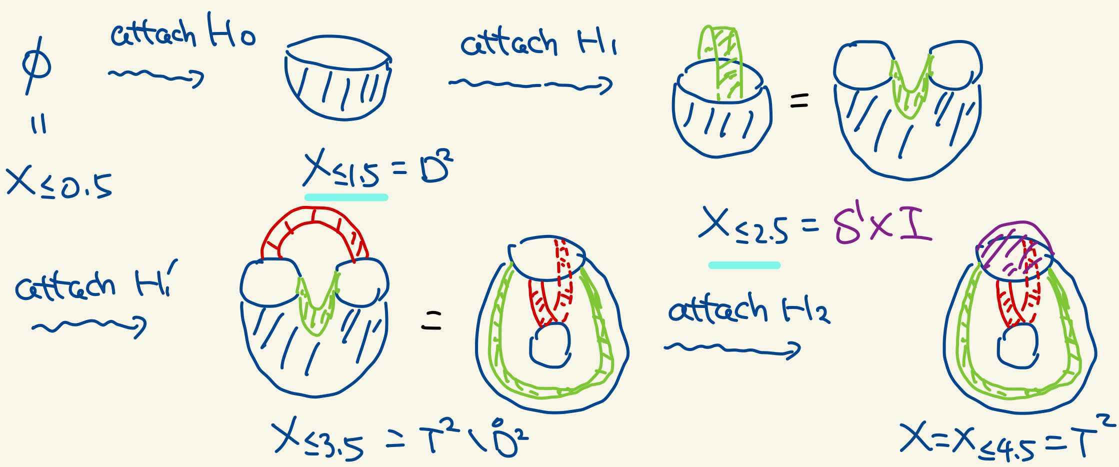
$$X_{\le r+\varepsilon} = X_{\le r-\varepsilon} \cup (H_{a_1} \sqcup H_{a_2} \cdots \sqcup H_{a_\ell})$$

In other words, we attach a k-handle on $X_{\le r}$ whenever we cross a critical point of index k. $\square$

Example:



Crit(f)	index
a_1	0
a_2	1
a_3	1
a_4	2



A handle decomposition of X (possibly with boundary) is a sequence $X_0, X_1, \dots, X_R$ s.t. X_R is a (disjoint) collection of handles, attached $(X_0 \cup X_1 \cdots \cup X_{R-1})$ and $X = X_0 \cup X_1 \cup \cdots \cup X_R$.

We write $X = X_0 | X_1 | X_2 \cdots | X_R$

Corollary: Any Morse function $f: X \to \mathbb{R}$ gives a handle decomposition of X.

Note: From any handle decomposition, one can also define a Morse function s.t. i-handles $\xleftrightarrow{1:1}$ index-i critical points.

Definition: A handle decomposion $X_0|X_1|\cdots|X_R$ is called monotone if X_i only consists of i-handles $\forall i$.

Proposition: Any manifold has a monotone handle decomposition.

Lemma Suppose $X = (Y \underset{\tilde{\varphi}_i}{\cup} H^i) \underset{\tilde{\varphi}_j}{\cup} H^j$ for some $i \geq j$

Then $\exists \tilde{\varphi}_j' : S^{j-1} \times D^{n-j} \to Y \setminus \text{image}(\tilde{\varphi}_i)$ s.t.

$$X \cong_{\text{diff}} (Y \underset{\tilde{\varphi}_j'}{\cup} H^j) \underset{\tilde{\varphi}_i}{\cup} H^i$$

Proof of Lemma: In $\partial(Y \underset{\tilde{\varphi}_i}{\cup} H^i)$, we have 2 spheres:

S_A^{j-1} : attaching sphere for H^j, S_B^{n-i-1} : belt sphere for H^i

$i \geq j \Rightarrow j-1+(n-i-1) < n-1 = \dim(\partial(Y \cup H^i))$ so we can isotope S_A^{j-1} s.t. $S_A^{j-1} \cap S_B^{n-i-1} = \emptyset$. Then $X \cong_{\text{diff}} Y \cup (H^i \sqcup H^j)$

$$= (Y \cup H^j) \cup H^i \quad \square$$

Proof of proposition: Use the above lemma to re-order the handles.

Definition: A Morse function $f: X \to \mathbb{R}$ is called "self-indexed" if $\forall x \in \mathrm{Crit}(f)$, we have $f(x) = \mathrm{index}(x)$.

Corollary: Every manifold has a self-indexed Morse function.

Idea: Handle decomposition is a "thickened version of CW decomposition" which always gives a manifold.

Actually, $(D^k \times D^{n-k}, S^k \times D^{n-k}) \xrightarrow{\text{retract}} (D^k, S^k)$

$X \xrightarrow{\text{homotopy equiv.}}$ a CW complex

Q: How to use handle decomposition to compute $H_*(X)$?

$X = X_0 | X_1 \cdots | X_n$ monotone

For $H_k \subset X_k$, $H_{k+1} \subset X_{k+1}$, we define

$\langle H_k, H_{k+1} \rangle := S_B^{n-k-1} \cdot S_A^k$ ← attaching sphere for H_{k+1}

↖ algebraic intersection in $\partial(X_0 | \cdots | X_k)$

belt sphere for H_k

Define $C_*^n(X)$ by $C_k^n(X) = \mathbb{Z}\langle \underbrace{H_k^1, \dots, H_k^{a_k}}_{k\text{-handles in } X_k} \rangle$

$d: C_k^n \to C_{k-1}^n$. $\quad dH_k^i := \sum_j \langle H_{k-1}^j, H_k^i \rangle \cdot H_{k-1}^j$

Theorem: $d^2 = 0$. $H_*(C_*^n(X)) = H_*(X)$. $\square$

Corollary (Morse inequality) For any $0 \le k \le \dim(X)$, we have

$$\sum_{0\le i\le k} (-1)^{k-i} b_i(X) \le \sum_{0\le i\le k} (-1)^{k-i} |\mathrm{Crit}_i(f)| \quad (\text{"="} \text{ if } k=\dim(X))$$

(This implies $|\mathrm{Crit}_i(f)| \ge b_i(X)$)

Corollary : Suppose all critical points have even indices

Then $b_{2k}(X) = \#$ critical points of index $2k$.

Handle moves

Theorem (Cerf) Any two monotone handle decompositions of X are related by the following handle moves

- handle slide
- creation/cancellation (• isotopy)

We say H_k, H_{k+1} form a $\boxed{\text{cancelling pair}}$ if

- $S_B^{n-k-1} \pitchfork S_A^k$ at a single point in $\partial(X_0|\cdots|X_k)$
- $S_A^k \cap S_{\alpha,B}^{n-k-1} = \emptyset \ \forall H_k^\alpha \neq H_k \subset X_k$.

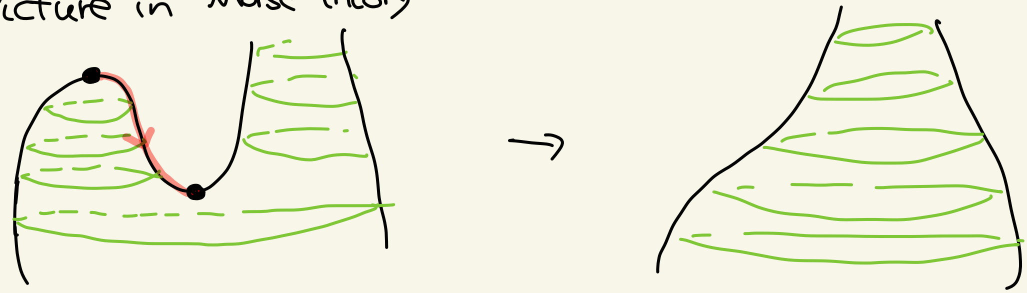
Lemma : Suppose H_k, H_{k+1} form a cancelling pair. Then

$$X_0|\cdots|X_k|X_{k+1}|\cdots|X_n \cong_{\text{diff}} X_0|\cdots|(X_k\setminus H_k)|(X_{k+1}\setminus H_{k+1})|\cdots$$

(This process is called cancellation of a handle pair.
The inverse process is called creation of a handle pair.)

Idea of proof: $H_k \cup H_{k+1} = D^n$, glued to $\partial(\cdots|X_{k-2})$ along D^{n-1}. This doesn't change diffeomorphism type.

Picture in Morse theory



• Handle slide

Let H_k^α H_k^β be 2 k-handles. The we have attaching spheres

$S^{k-1}_{\alpha,A}, S^{k-1}_{\beta,A} \hookrightarrow \partial(X_0|\cdots|X_{k-1}) = Y$ both framed.

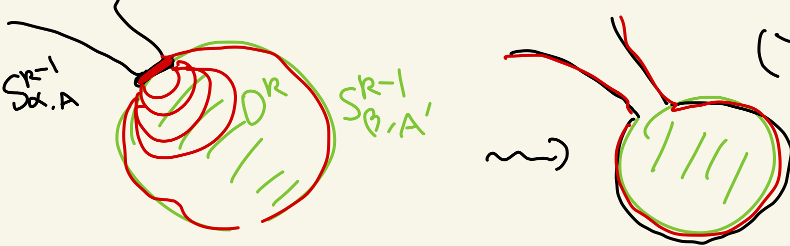
Using the framing, we can "push" $S^{k-1}_{\beta,A}$ a little and get $S^{k-1}_{\beta,A'}$

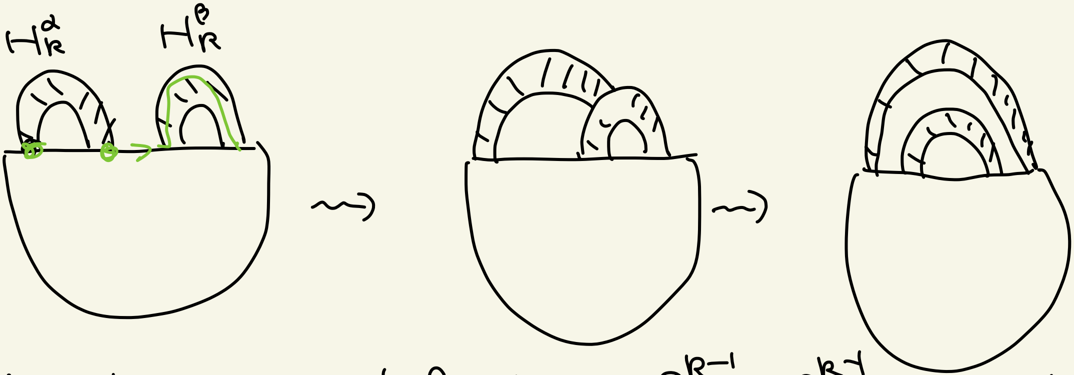
Handle slide of H_k^α along H_k^β : $S^{k-1}_{\alpha,A} \rightsquigarrow S^{k-1}_{\alpha,A} \# S^{k-1}_{\beta,A'}$

Keep $S^{k-1}_{\beta,A}$ and other attaching spheres.

This correspond to isotoping $S^{k-1}_{\alpha,A}$ in $\partial(X_0|\cdots|X_{k-1}|H_k^\beta)$

$$= (Y \setminus \nu(S^{k-1}_{\beta,A})) \cup (\underbrace{D^k}_{\text{bounded by } S^{k-1}_{\beta,A'}} \times S^{n-k})$$





Will talk more about framing on $S^{k-1}_{A,\alpha} \# S^{k-1}_{A',\beta}$ ~~next lecture~~.

Proof of the h-cobordism theorem.

Theorem (Smale): W is a smooth h-cobordism from X (dimension n) to X'

Assume $\pi_1 X = 1$. $n \geq 5$. Then $W \cong_{diff} X \times [0,1]$

Proof: Take a self-indexed Morse function

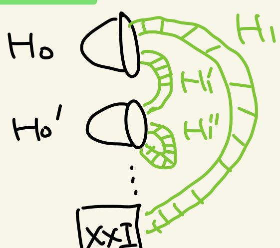
$$f: W \to \mathbb{R} \quad \text{s.t.} \quad f^{-1}(-1) = X \quad f^{-1}(n+1) = X'$$

This gives a handle decomposition of the cobordism

$$W = X \times [0,1] \,|\, X_0 \,|\, X_1 \,-\!-\!-\, |\, X_{n+1}$$

$\boxed{\text{Goal}}$: eliminate all handles

Step 1: Cancel 0-handles with 1-handles



Every H_1 is attached to $\partial(X \times [0,1] \,|\, X_0)$

along $S^0 \times D^n = D^n \sqcup D^n$ two feet

If exactly one of them $\subset H_0$

then H_1, H_0 form a cancelling pair

W connected $\Rightarrow X \times I \,|\, X_0 \,|\, X_1$ is connected $\Rightarrow$ all H_0 can be cancelled

Similary, we can cancel all $(n+1)$-handles with some n-handles

($-f \Rightarrow$ "dual" handle decomposition)

so $W = (X \times I) \,|\, X_1 \,|\, X_2 \,|\, \cdots \,|\, X_n$

Step 2 : "trade 1-handles with 3-handles"

$\pi_1(W)=0 \Rightarrow \pi_1(X\times I \mid x_1 \mid X_2)=0$

For every H_1, we can do a handle creation $H_2 \sqcup H_3$ s.t. H_2, H_1 can be cancelled. (See Milnor's book for detail)

Similarly, we can trade n-handles with $(n-2)$-handles

$$W = X\times[0,1] \mid X_2 \mid \cdots \mid X_{n-1}$$

Step 3 : Cancel

$C_k^n(W) := \mathbb{Z}\langle k\text{-handles}\rangle$, $d H_k^\alpha = \sum_\beta \langle H_k^\alpha, H_{k-1}^\beta\rangle H_{k-1}^\beta$

$H_*(C_k^n(W)) = H_*(W, X) = 0$

$$\cdots \xrightarrow{d_3} C_3^n \xrightarrow{d_2} C_2^n \to 0$$

$H_2(C_*^n) = 0 \Rightarrow d_2$ surjective $\qquad dH_3 = H_2$

For any $H_2 \in C_2^n$, $\exists \sum a_i H_3^i$ s.t. $d(\sum a_i H_3^i) = H_2$

By doing handle slide $\sum a_i H_3^i \rightsquigarrow$ single 3-handle H_3

so we have $\langle H_3, H_2\rangle = 1 \qquad \langle H_3, H_2^\alpha\rangle = 0 \ \forall H_2^\alpha \neq H_2$

I.e. $S_A^2 \cdot S_B^{n-2} = 1 \qquad S_A^2 \cdot S_{\alpha,B}^{n-2} = 0$

$\pi_1(\partial(X\times I \mid X_2)) = 1$ $\dim \geq 5$. So we apply Whitney's lemma and isotope S_A^2 s.t. $S_A^2 \pitchfork S_B^{n-2}$ at single point and $S_A^2 \cap S_{\alpha,B}^{n-2} = \emptyset$. So can cancel H_3 with H_2.

Keep doing this and cancels all 2-handles

3-handles, - - - -