Lecture 6. Alexander's theorem and Markov's theorem

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June, 2, 2022

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Braid closure

Definition 1.1

The closure of a braid b is the link Cl(b) obtained from b by connecting the lower ends of the braid with the upper ends; see Fig. 1.

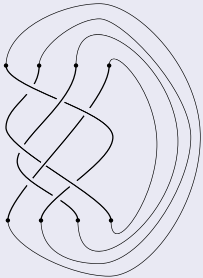

Figure 1: A braid closure

Obviously, isotopic braids generate isotopic links.

Remark 1.2

Closures of braids are usually taken to be oriented: all strands of the braid are oriented from the top to the bottom.

Some braids generate knots and the others generate links. In order to calculate the number of components of the corresponding link, one should take into account the following simple observation. In fact, there exists a simple natural epimorphism from the braid group onto the permutation group $\Sigma\colon \mathrm{Br}(n) \to S_n$, defined by $\sigma_i \to s_i$, where s_i are natural generators of the permutation group.
Consider a braid B. Obviously, for all numbers p belonging to the same orbit of the natural permutation action (of $\Sigma(B)$) on the set $\{1, \ldots, n\}$, all upper vertices with abscissas $(p, 0)$ belong to the same link component.
Consequently, we obtain the following proposition.

Proposition 1.3

The number of link components of the link of the closure $\mathrm{Cl}(B)$ equals the number of orbits of action for $\Sigma(B)$.

Obviously, non–isotopic braids might generate isotopic links. We will touch on this question later.

An interesting question is to define the minimal number of strands of a braid whose closure represents the given link isotopy class L. Denote this number by Braid(L).

An interesting theorem on this theme belongs to Birman and Menasco.

Theorem 1.4

For any knots K_1 and K_2, the following equality holds:

$$\mathrm{Braid}(K_1 \# K_2) = \mathrm{Braid}(K_1) + \mathrm{Braid}(K_2) - 1.$$

In Fig. 2 we show that if the knot K_1 can be represented by an n–strand braid, and K_2 can be represented by an m–strand braid, then $K_1\#K_2$ can be represented by an $(n+m-1)$–strand braid. This proves the inequality "$\leq$".

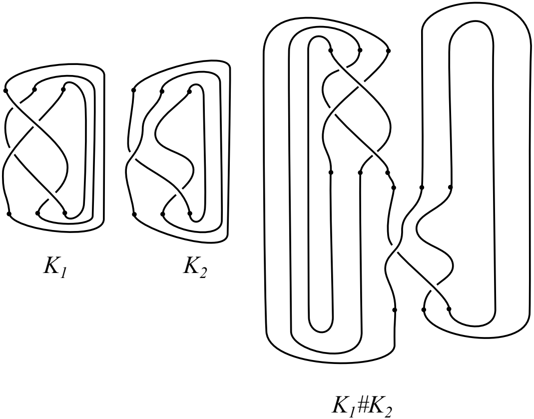

Figure 2: Representing the connected sum of braids

The proof for the inequality "$\geq$" is due to Birman and Menasco.

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The Alexander theorem

Theorem 1.5 (Alexander theorem)

Each link can be represented as the closure of a braid.

We shall give two proofs of this theorem: the original one by Alexander and the one by Vogel that realises a faster algorithm for constructing a corresponding braid.

Definition 1.6

A polygonal link is a link, which consists of line segments as Fig. 3.

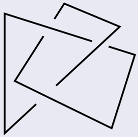

Figure 3: A polygonal link

Alexander's proof

We shall prove this theorem for the case of polygonal links. Consider a diagram L of an oriented polygonal link and a point O on the plane P of the diagram (this point should not belong to edges and should not coincide with vertices of the diagram). We say that L is braided around O if each edge of L is visible from O as counterclockwise–oriented.

Definition 1.7

For any L and O, let us call edges visible as counterclockwise–oriented positive; the other ones will be negative.

Alexander's proof (continued)

If there exists a point O such that our link diagram is braided around O, then the statement of the Alexander theorem becomes quite clear: we just cut the diagram along a ray coming from O and "straighten the diagram"; see Fig. 4.

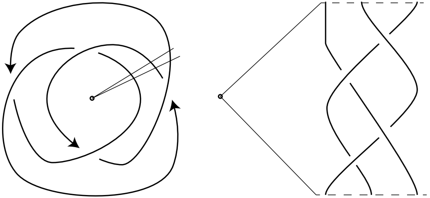

Figure 4: Constructing a braid by a braided link

Thus, in order to prove the general case of the theorem, we shall reconstruct our arbitrary link diagram in order to obtain a diagram braided around some point O.

Alexander's proof : Alexander trick

First, fix a point O. Now, we are going to use the Alexander trick as follows. Consider a negative edge AB of our polygonal link and find some point C on the projection plane P such that the triangle ABC contains O. Then we replace AB by AC and CB. Both edges will evidently be positive; see Fig. 5.

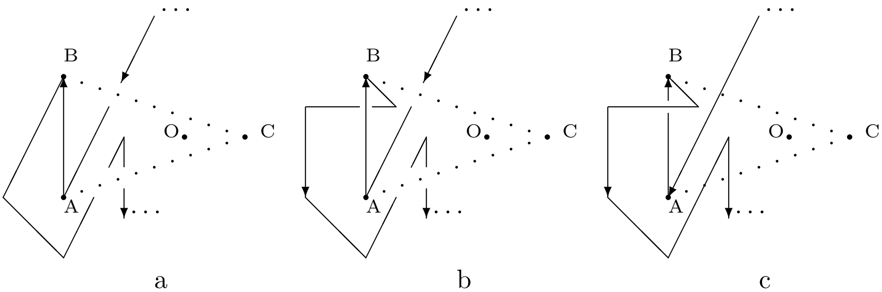

Figure 5: Alexander's trick

Alexander's proof : Alexander trick

We shall use this operation till we get a diagram braided around O and the proof is completed.

Let us describe this construction in more detail. In the case when the negative edge AB contains no crossings, the Alexander trick can be easily performed directly; see Fig. 5.a. Actually, one can divide the edge AB into two parts (edges) and then push them over O see Fig. 6.

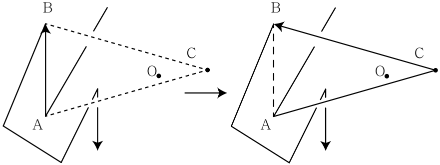

Figure 6: Case 5.a.

Alexander's proof : Alexander trick (continued)

The same can be done in the case when AB contains the only crossing that is an overcrossing with respect to the other edge; see Fig. 5.b. Finally, if AB contains the only crossing that is an undercrossing with respect to the other edge, then we can push it under, as shown in Fig. 5.c.

Remark 1.8

The method of proof for the Alexander theorem described above certainly gives us a concrete algorithm for constructing a braid from a link. However, this algorithm is too slow. Below, we give a simpler algorithm for constructing braids by links.

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Vogel's algorithm

Here we describe the algorithm proposed by Pierre Vogel.
We begin with a definition.

Definition 1.9

An oriented link diagram is braided if there exists a point on the plane of the diagram around which the link diagram is braided.

A braided link diagram can be easily represented as a closure of a braid.

Remark 1.10

Obviously, the property of a diagram to be braided does not depend on the crossing structure. We may say that we shall work only with shadows of links. In the sequel, we shall never use this structure.

Given an oriented closed diagram D of a link L, one can correctly define the operation of crossing smoothing for it. To do it, we just "smooth" the diagram at each vertex as shown in Fig. 7 and consider all Seifert circles of it. Denote this smoothing by σ.

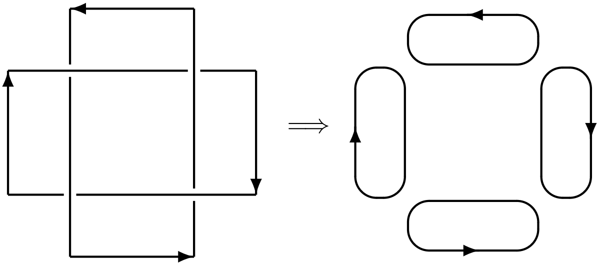


Figure 7: Smoothing of crossings and Seifert circles

Definition 1.11

Let us say that all Seifert circles of some planar diagram are nested if they all induce the same orientation of the plane and bound an enclosed disc system.

Obviously, if all Seifert circles of some planar link diagram are nested, then the corresponding diagram is braided. Moreover, in this case, the number of strands of the braid coincides with the number of Seifert circles.

Let us fix some link diagram D and consider now the shadow of D. This shadow divides the sphere (one–point compactification of the plane) into 2–cells, called sides. The interior side is that containing infinity.

Definition 1.12

A side S is unordered if it has two edges a, b that belong to different Seifert circles A_1, A_2 and induce the same orientation of S, and ordered otherwise.

In the first case we say that the edges A_1, A_2 generate the unordered side.
One can apply the move Ω_2 to the edges A_1, A_2, which generate the unordered side, as shown in Fig. 8. In this case, the set of sides becomes "more ordered".

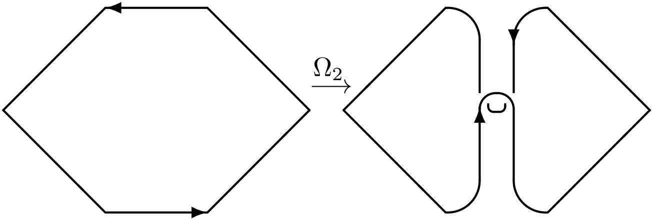


Figure 8: Reduction for a pair of unordered sides

Proposition 1.13

If all edges of the side Σ belong to two Seifert circles then this side is ordered.

Proof.

Actually, consider the edges of this side. It is easy to see that all edges belonging to the same Seifert circle have the same orientation.

Consider two adjacent edges belonging to different Seifert circles. They have different orientations. Thus, any two edges of the given diagram belonging to different Seifert circles must have different orientations. Hence, the side is ordered.

Proposition 1.14

If a diagram D of the link L has no unordered sides, then it can be transformed to a braided diagram by using an infinity change.

Proof Suppose the diagram D has no unordered sides. Consider some side of the planar tiling generated by our link diagram. Any two adjacent edges of this side either have the same orientation (in this case, they belong to the same Seifert circle) or they have different orientations (and belong to different Seifert circles). If we consider the points of adjacent edges belonging to the same Seifert circle as the points of one "long" edge then we obtain some polygon M (or a whole Seifert circle).

Proof (continued)

The edge orientations of the edges of M are alternating. Thus, the number of such edges is even (or equal to one when all edges belong to the same Seifert circle). Since this side is not unordered, all edges of it belong to no more than two different Seifert circles. Thus we conclude that each Seifert circle that defines some edge of the polygon M is adjacent either to one Seifert circle or to two Seifert circles (lying on different sides of M). Otherwise, there would be an unordered side with edges belonging to three different Seifert circles. The remaining part of the proof is left as a simple exercise.

Vogel algorithm (continued)

The Vogel algorithm works as follows. First we eliminate all crossings by the rule:  $\to$ ,  $\to$ . Then, by using Ω_2 we remove unordered sides. Finally, if Seifert circles are not nested, we change the infinity.

Let us describe this algorithm in more detail.

First, let us smooth all crossings of the diagram. Thus we obtain several Seifert circles. Denote the number of these circles by s. Some pairs of these circles might generate unordered sides. Let us construct a graph whose vertices are Seifert circles; two vertices should be connected by an edge if there exists a side (ordered or not) that is incident to the two circles. Let us remove from this graph a vertex, corresponding to some "interior" Seifert circle. We obtain some graph Γ_1. Let us change the notation for the remaining $s - 1$ circles and denote them by $A_1, A_2, \ldots, A_{s-1}$, in such a way that A_i and A_{i+1} contain edges that generate an unordered side. This means that our graph Γ_1 is connected. It is easy to see that in the disconnected case we should apply this algorithm to each connected component; it will work even faster.

Vogel algorithm (continued)

We shall perform the following operation. Let us take the unordered side, generated by A_1 and A_2, and perform Ω_2 to it as described above. Instead of circles A_1 and A_2, we shall get two Seifert circles; one of them lies inside the other. Besides this, they do not generate an unordered side; see Fig. 9.

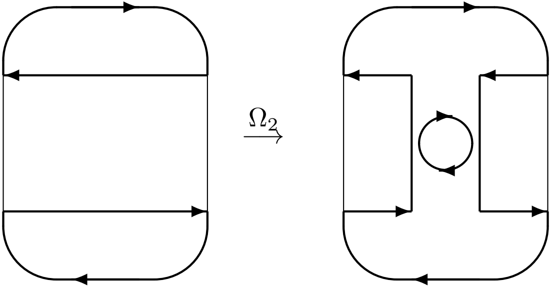


Figure 9: Eliminating an unordered side

Vogel algorithm (continued)

We have got two new circles; one of them lies inside the other. Denote the exterior circle by A_1 and the interior one by A_2. Because "the former A_2" generated an unordered side together with A_3 then the new circle A_1 also generates an unordered side together with A_3 (the latter stays the same).

Let us now perform Ω_2 on the circles A_1 and A_3 and change the notation again: the exterior circle will be A_1 and the interior one will be A_3, and so on. Finally (after $s-2$ operations Ω_2), we obtain one interior circle A_1 that makes no unordered sides. Now, we shall not touch A_1, but perform the same procedure with the pairs (A_2, A_3), (A_2, A_4), and so on. Then we do the same for $A_3, A_i, i > 3$, and so forth. Thus, we have performed $\frac{(s-1)(s-2)}{2}$ second Reidemeister moves and (possibly) one infinity change and obtained the set of circles $A_1, A_2, \ldots, A_{s-1}$, where each next circle lies inside all previous ones and no two circles generate an unordered side.

Let us show that the remaining circle (that we "removed" in the very beginning) does not make unordered sides either.

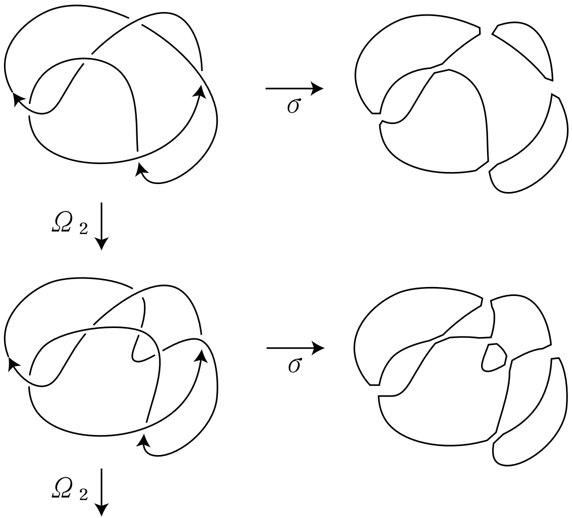
Actually, since this circle has some exterior edge it could generate an unordered side only with A_1, but this is not the case. After this, we should change the infinity (if necessary). Thus, after C^2_{s-2} operations (for the connected case; in the unconnected case we shall use even fewer operations) we obtain a braided diagram.
So, we have proved the following.

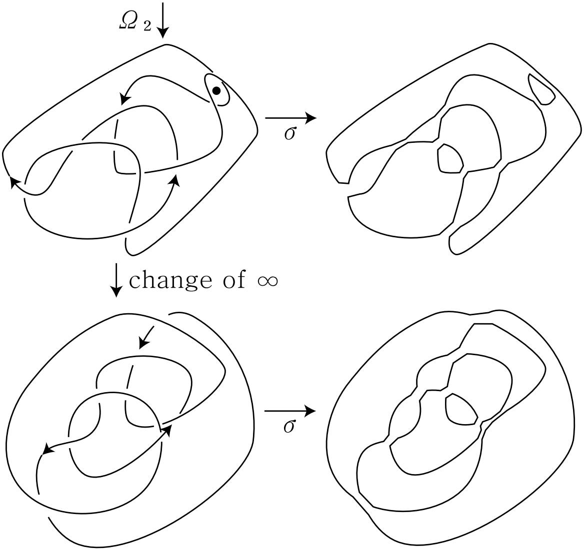
Theorem 1.15

If the link diagram D has n crossings and s Seifert circles then

1. The Vogel algorithm requires no more than C^2_{s-2} second Reidemeister moves.
2. The total number of strands of the obtained braid equals s and the number of crossings does not exceed $n + (s-1)(s-2)$.

In the next two pages, we perform the Vogel algorithm for the knot named 5_2.


σ
Ω 2
σ
Ω 2


Ω_2
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Plat closure

For braids $\beta \in \mathrm{Br}(2n)$ there is another way to obtain a link, called plat closure as shown in Fig. 10.

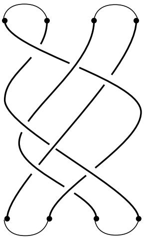

Figure 10: Plat closure

Birman proved a kind of Alexander theorem for plat closures.

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Introduction: Markov's theorem

In the present section, we shall give another proof of Alexander's theorem and a proof of Markov's theorem by using L-moves. Originally L-moves appeared in the paper by Lambropoulou and Rourke [LR], and then they were very intensively explored for virtual braids and many other braid groups (Lambropoulou, Kauffman and others).

Let us now formulate the Markov's theorem:

Theorem 2.1 (Markov's theorem)

The closures of braids A and B are isotopic if and only if B can be obtained from A by a sequence of the following moves (Markov moves):

1. conjugation $b \to a^{-1}ba$ by an arbitrary braid a with the same number of strands as b,
2. the move $b \to b\sigma_n^{\pm 1}$, where b is a braid on n strands and the obtained braid has $n + 1$ strands,
3. the inverse transformation of 2.

Markov moves

The necessity of these two moves is evident. The isotopies between corresponding pairs of braid closures are shown in Fig. 11.
In Fig. 11.b the first Reidemeister move comes into play. This move did not take part in braid isotopies, so this kind of knot isotopy appears here.

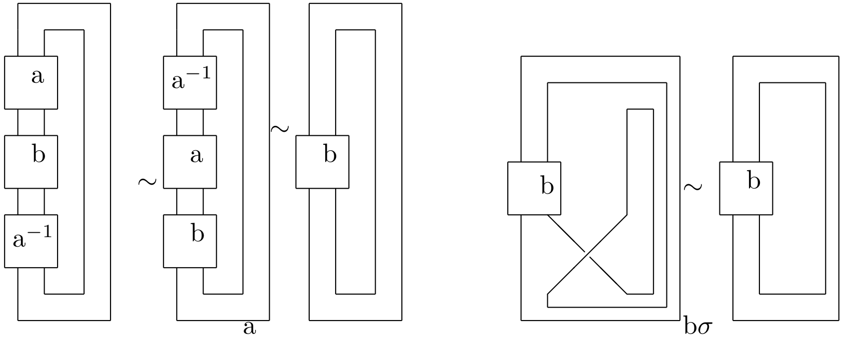

Figure 11: Two Markov moves represent isotopy

Main idea

The main idea of the L-move method of the proof of Markov's theorem is as follows. We try to make a braid with infinite ends. Good news is that one can construct a braid diagram out of link hence proving the Alexander theorem and then, when analysing the ambiguity in construction and the effect of Reidemeister's moves on the initial braid diagram, one gets the Markov moves.

L-move on braids: definition

Definition 2.2

An L-move on a braid consists of cutting one arc of the braid open and splicing into the broken strand new strands to the top and bottom, both either under or over the rest of the braid

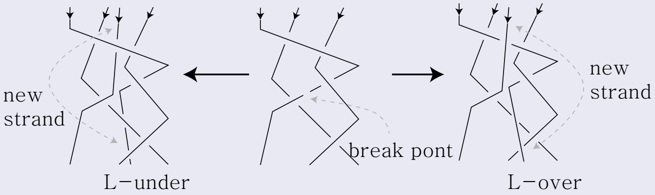

Figure 12: L-moves

L-moves and isotopy generate an equivalence on braids called L-equivalence. Our goal is to show that L-equivalence classes of braids are in bijective correspondence with isotopy classes of oriented links in S^3 where the bijection is induced by "closing" the braid to form a link. As a consequence, L-equivalence is the same as the usual Markov equivalence, and thus we get the classical Markov's theorem.

The proofs are based on a canonical process for turning a combinatorial oriented link diagram in the plane (with a little extra structure) into an open braid. Our braiding as well as the L-moves are based on building blocks of combinatorial isotopy, the triangle moves or Δ-moves.
We use the point at the infinity as the reference point for braiding. The change of reference point to infinity makes the proof of the Markov theorem easy.

The starting point

There are two combinatorial moves on diagrams which we shall consider:

1. Δ-move: An arc is replaced by wo arcs forming a triangle (and its inverse) respecting orientation and crossings. "Respecting crossings" mean that, if we lift the diagram to an embedding in 3-space, then the Δ-move lifts to an elementary isotopy move
2. Subdivision: A vertex is introduced/deleted in an arc of the diagram.

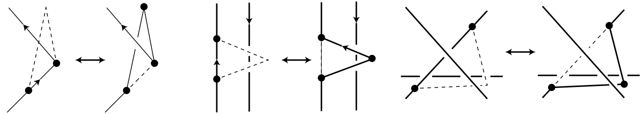

Figure 13: Reidemeister moves

Subdivision

Subdivision moves can be viewed as special cases of Δ-moves. We shall call the equivalence relation generated by these two moves a combinatorial isotopy or just an isotopy.
As we know, this notion of isotopy is equivalent to the standard one when we pass to the smooth category. We left the verification of this statement as an exercise.

Definition of L-moves

Definition 2.3 (L-moves.)

Let D be a link diagram/braid and P a point of an arc of D such that P is not vertically aligned with any of the crossings or (other) vertices of D (note that P itself may be a vertex). Then we can perform the following operation: Cut the arc at P, bend the two resulting smaller arcs apart slightly by a small isotopy and introduce two new vertical arcs to new top and bottom end-points in the same vertical line as P. The new arcs are both oriented downwards and they run either both under or both over all other arcs of the diagram. Thus there are two types of L-moves, an under L-move or L_u-move and an over L-move or L_o-move.

This gives the following algebraic expression for an L_o-move and an L_u-move respectively.

$$\alpha = \alpha_1\alpha_2 \sim \sigma_i^{-1} \ldots \sigma_n^{-1}\tilde{\alpha_1}\sigma_{i-1}^{-1} \ldots \sigma_{n-1}^{-1} \ldots \sigma_i\tilde{\alpha_2}\sigma_n \ldots \sigma_i$$

$$\alpha = \alpha_1\alpha_2 \sim \sigma_i \ldots \sigma_n\tilde{\alpha_1}\sigma_{i-1} \ldots \sigma_{n-1} \ldots \sigma_i^{-1}\tilde{\alpha_2}\sigma_n^{-1} \ldots \sigma_i^{-1}$$

where α_1, α_2 are elements of B_n and $\tilde{\alpha_1}, \tilde{\alpha_2} \in B_{n+1}$ are obtained from α_1, α_2 by replacing each σ_j by σ_{j+1} for $j = i, \ldots, n-1$.

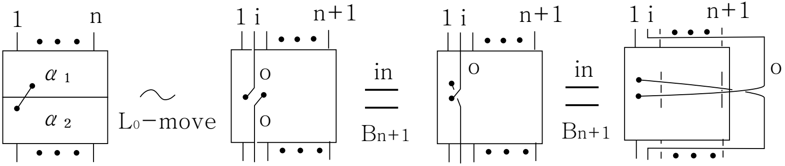

Figure 14:

One-move Markov's theorem

Theorem 2.4

The closure of braids induces a bijection between the set of L-equivalence classes of bradis and the set of isotopy types of (oriented) link diagrams.

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The braiding process

We shall define an inverse bijection to $\mathcal{C}$ by means of a canonical braiding process which turns an oriented link diagram (with little extra structure) into a braid. Note that we only work with oriented diagrams. Let D be a link diagram with no horizontal arcs, and consider the arcs in D which slope upwards with respect to their orientations; call these arcs opposite arcs. In order to obtain a braid from that diagram we want:

1. to keep the arcs that go downwards;
2. to eliminate the opposite arcs and produce braid strands instead.

The braiding process: continuation

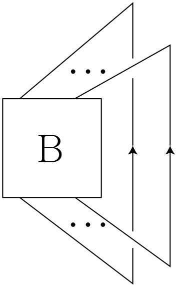

Figure 15: The closure of B

Triangle condition

Definition 2.5 (Triangle condition)

Non-adjacent sliding triangles are only allowed to meet if they are of opposite types (i.e. one over and the other under).

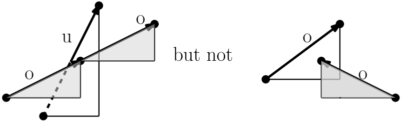

Figure 16: Triangle condition

Lemma 2.6

Given a link diagram D, there is a subdivision D of D such that (for appropriate choices of under/over for free up-arcs) the triangle condition is satisfied.

Generic diagram and generic Δ-move

Definition 2.7

A generic diagram is a link diagram with subdividing points and sliding triangles put in general position with respect to the height function, such that the following conditions hold:

1. there are no horizontal arcs,
2. no two disjoint subdividing points are in vertical alignment, where by 'disjoint' we mean subdividing points that do not share a common edge.
3. any two non-adjacent sliding triangles satisfy the triangle condition and if they intersect, this should be along a common interior (and not a single point).

Definition 2.8

A generic Δ-move is a Δ-move between generic diagrams.

Lemma 2.9

An isotopy between generic link diagrams can be realised using only generic Δ-moves.

Proof. If after some Δ-move during the isotopy appears a horizontal arc or vertical alignment of vertices we remove it by replacing one of the participating vertices by a point arbitrarily close to it, so that the new point will not cause such a singularity or violation of condition 3 in all the diagrams of the isotopy chain. If two non-adjacent sliding triangles touch so that condition 3 is violated (Fig. 17) we argue as above.

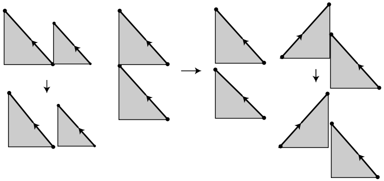

Figure 17:

Proof of Lemma 2.9 (continued)

The remaining possibilities are the ones illustrated in Fig. 18 and they can be removed. The proof is completed.

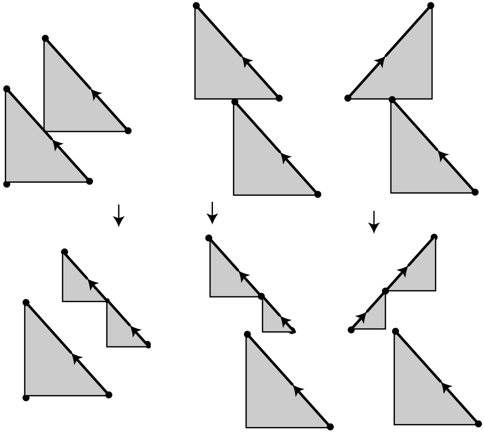

Figure 18:

Alexander's theorem

Corollary 2.10 (Alexander's theorem)

Any (oriented) link diagram is isotopic to the closure of a braid.

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By the local nature of the Δ-moves we may assume that for a given link diagram we have done the braiding for all up-arcs except for the ones that we are interested in every time; these will be lying in the "magnified" region placed inside the braid

Now, two braids that differ by a finite sequence of L-moves have isotopic closures. Therefore the function C from L-equivalence classes of braids to isotopy types of link diagrams is well-defined. To show that C is a bijection we shall use our braiding process to define an inverse function B. Namely, for a diagram D let B(D) be the braid resulting from the braiding algorithm applied to it. We have to show that B is a well-defined function from link-diagram types to L-equivalence classes of braids, therefore we have to check that B(D) does not depend up to L-equivalence on the choices made before the braiding and on Δ-moves between link diagrams.

Lemma 2.11

If we add on an up-arc, α, an extra subdividing point P and label the two new up-arcs, α_1 and α_2, the same as α, the corresponding braids are L-equivalent.

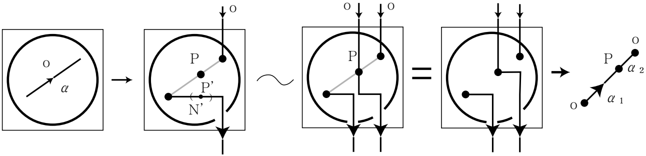

Figure 19:

Lemma 2.12

When we meet a free up-arc, which we have the choice of labelling 'u' or 'o', the resulting braid does not depend – up to L-equivalence – on this choice.

Corollary 2.13

If we have a chain of overlapping sliding triangles of free up–arcs so that we have a free choice of labelling for the whole chain then, by Lemmas 2.11 and 2.12, this choice does not affect – up to L-equivalence – the final braid.

Corollary 2.14

If by adding a subdividing point on an up–arc we have a choice for relabelling the resulting new up–arcs so that the triangle condition is still satisfied then, by Lemmas 2.11 and 2.12,the resulting braids are

Corollary 2.15

Given any two subdivisions, S_1 and S_2, of a diagram which will satisfy the triangle condition with appropriate labellings, the resulting braids are L-equivalent.

Exercises

1. Construct braids whose closures represent the following links by using "Alexander trick"
   - The left- and right- handed trefoils,
   - The figure eight knot,
   - the Borromean rings.
2. Construct braids whose closures represent the following links by using "Vogel algorithm"
   - The left- and right- handed trefoils,
   - The figure eight knot,
   - the Borromean rings.
3. Show that closures of two n–strand braids are isotopic in the class of closures of n–strand braids if and only if these two braids are conjugated.
4. Perform Markov moves and braid isotopy to show the following two torus knots are equivalent:
   - $T(2, 2n + 1)$ and $T(2n + 1, 2)$,
   - $T(3, 4), T(4, 3)$,
   - $T(p, q), T(q, p)$.

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