

# Quadriseccants of knots and links

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We show that every non-trivial tame knot or link in  $\mathbb{R}^3$  has a quadriseccant, i.e. four collinear points. The quadriseccant must be topologically non-trivial in a precise sense. As an application, we show that a nonsingular, algebraic surface in  $\mathbb{R}^3$  which is a knotted torus must have degree at least eight.

## 1. INTRODUCTION

An elementary count of degrees of freedom suggests that a randomly-chosen curve in  $\mathbb{R}^3$ , if sufficiently complicated, should contain four collinear points. One precise interpretation of this intuition is the following two theorems:

**Theorem 1.1 (Pannwitz, Morton, Mond).** *Every non-trivial piecewise linear or smooth knot in  $\mathbb{R}^3$  in general position has four collinear points.*

**Theorem 1.2 (Pannwitz, Morton, Mond).** *If two smooth or PL circles  $A$  and  $B$  in  $\mathbb{R}^3$  in general position have a non-zero linking number, then there is a line in  $\mathbb{R}^3$  which intersects  $A$ , then  $B$ , then  $A$  again, and then  $B$  again.*

These theorems are presented in [5] and [6]. (They are also mentioned in [2].) Also, the arguments in [6] yield a lower bound on the number of collinearities and a generalization of the second theorem to the case of two circles which are linked in the sense that each represents a non-trivial homotopy class in the complement of the other. The main theorem of the present paper is a different generalization of this result:

**Theorem 1.3.** *Every non-trivial tame link in  $\mathbb{R}^3$  has four collinear points.*

Since the statement of the theorem resembles the statements of theorems of Pannwitz, Morton, and Mond, we explain the extra cases covered by our theorem. By a *non-trivial link* we mean any set of disjoint circles embedded in  $\mathbb{R}^3$  such that there is no homeomorphism of  $\mathbb{R}^3$  which sends the circles to a flat plane. The Whitehead link and the Borromean rings are two examples of non-trivial links which are not covered by the previous theorems. A *tame link* is any set of continuous circles which are collared by solid tori, or equivalently one which is topologically equivalent to a smooth link in  $\mathbb{R}^3$ . However, a tame link may be very different from a smooth link geometrically; for example, its Hausdorff dimension may be greater than 1. Moreover, the main theorem is not restricted to links which have any particular transversality properties or are in general position in any sense.

To eliminate the general position hypothesis, we first prove a stronger theorem about (smooth) links in general position: Such a link has a line which intersects it four times in a topologically non-trivial way. Armed with this extra condition,

we can use a limiting argument to pass from links in general position to arbitrary tame links.

The theorem has an interesting corollary which may be applied to the topology of real algebraic surfaces. It is this application which led the author to the topic of this paper.

**Corollary 1.4.** *If an algebraic surface in  $\mathbb{R}^3$  contains the boundary of a knotted solid torus or linked solid tori, the surface has degree at least 8.*

The theorem inspires a definition:

**Definition 1.5.** *If  $L$  is a link in  $\mathbb{R}^3$ , a secant of  $L$  is a line segment whose endpoints lie in  $L$ , a triseccant of  $L$  is a secant of  $L$  and a point  $p$ , the middle point, which lies in both  $L$  and the interior of the secant, and a quadriseccant is a secant with two middle points.*

To be precise, a quadriseccant is a pair of distinct triseccants with the same underlying line segment. A *degenerate secant* is a single point. The set of secants has a natural topology, as does the set of triseccants: for a sequence of triseccants to converge we insist that the middle points converge as well.

As motivation for the main theorem, we present a simple proof of a weaker result:

**Theorem 1.6.** *Every non-trivial smooth knot  $K$  in  $\mathbb{R}^3$  has a triseccant.*

*Proof.* Suppose that there exists a point  $p$  in  $K$  such that no points  $q$  and  $r$  in  $K$  are collinear with  $p$ . Then the union of the chords  $\overline{pq}$  for all  $q$  in  $K$  is evidently a smooth embedded disk with boundary  $K$ , which renders  $K$  trivial.  $\square$

This proof illustrates the central idea in the proof of the main theorem.

I would like to thank my advisor, Andrew Casson, for encouragement and helpful comments.

## 2. GENERAL POSITION

There is a general theory of general position, presented in a paper by Wall [9] and used in [5]. We review the elements of this theory needed here:

**Definition 2.1.** *If  $X$  is a topological space with a measure, a property  $P$  of members of  $X$  is generic if it is true on a set with full measure, and a member of  $X$  is in general position with respect to  $P$  if it satisfies  $P$ . A member of  $X$  is in general position if it is in general position with respect to all applicable generic properties mentioned in this paper.*

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Usually  $X$  is a space of functions. We define a *polynomial function* from the unit circle  $S^1$  in  $\mathbb{R}^2$  to  $\mathbb{R}^3$  to be a function which is given by polynomials of some degree  $d$  in the standard coordinates in  $\mathbb{R}^2$ . The set of all such functions forms a finite-dimensional vector space  $P_d$ , and we consider all generic properties relative to  $P_d$  for some  $d > 0$  with the usual Cartesian topology and measure. A function  $K : S^1 \rightarrow \mathbb{R}^3$  is a *knot* if it is injective, and since this is a generic property, we are justified in referring to polynomial functions in general position as polynomial knots.

More generally, we may define  $k(S^1)$  to be the disjoint union of  $k$  unit circles, consider the vector space  $P_{d,k}$  of  $k$ -tuples of polynomial functions, and define a *link* to be an injective function from  $k(S^1) \rightarrow \mathbb{R}^3$  for some  $k$ .

The concept of a polynomial link is not an essential ingredient in this paper, but the following lemmas, whose proofs are easy, make it a useful one:

**Lemma 2.2.** *Given an arbitrary smooth function  $f : k(S^1) \rightarrow \mathbb{R}^3$ , there is a sequence of polynomial links (of varying degree) whose values and first derivatives converge uniformly to those of  $f$ . We may choose the sequence to be in general position.*

**Lemma 2.3.** *A property  $P$  of members of a finite-dimensional vector space is a polynomial property or an algebraically generic property if there exists some non-trivial polynomial  $p$  on the vector space such that  $P$  is true at all points for which  $p$  is non-zero. All polynomial properties are generic.*

If  $L$  is a polynomial link with  $k$  components, we define a projection function  $\pi_L : k(S^1) \times k(S^1) - \Delta \rightarrow S^2$ , where  $\Delta$  is the diagonal, by:

$$\pi_L(a, b) = (L(a) - L(b)) / |L(a) - L(b)|.$$

We view  $\pi_L$  as a family of maps  $\pi_L(\cdot, b)$  parameterized by the second variable.

The main result of this section is the following lemma. Neither the lemma nor the proof have more mathematical content than equivalent lemmas in [5] and [6], and the key idea is originally due to Reidemeister [7], so the proof here is sketched to some extent.

**Lemma 2.4.** *With  $L$  and  $\pi_L$  defined as above, it is a polynomial property for  $L$  to be a smooth embedding, i.e. its derivative does not vanish anywhere. It is also a polynomial property of  $L$  for there to exist a finite set of points of  $k(S^1)$ , called the set of special points, whose complement is the set of generic points, such that for a generic point  $a$  and a special point  $b$ :*

- I.  $\pi_L(\cdot, a)$  is a smooth immersion of a 1-manifold with ends, where the ends correspond to the tangent directions of  $L$  at  $a$ .
- II.  $\pi_L(\cdot, a)$  does not pass through the two tangent directions.
- III.  $\pi_L(\cdot, a)$  is everywhere one-to-one or two-to-one.
- IV. If  $\pi_L(\cdot, a)$  is two-to-one at a point of  $S^2$ , it is self-transverse at that point.

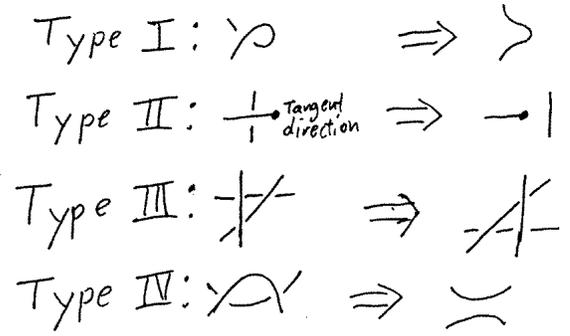


Figure 1

- V.  $\pi_L(\cdot, b)$  has all of the previous properties at all but one point of  $S^2$  and has three of the previous properties at the remaining point  $p$ . In this case, as  $a$  varies from one side of  $b$  to the other, the structure of  $\pi_L(\cdot, a)$  near  $p$  is characterized by one of the corresponding diagrams in Figure 1.

*Proof.* We define the *algebraic dimension* of a subset  $S$  of a vector space  $V$  to be the Krull dimension of the ring of polynomial functions restricted to  $S$ . (The Krull dimension of a commutative ring is the maximum length of an ascending chain of prime ideals [3].) We will need two basic facts about algebraic dimension: The algebraic dimension image of a set  $S$  under a projection (or more generally a polynomial map) is less than or equal to the algebraic dimension of  $S$ , and the complement of a set of algebraic codimension 1 or more is a polynomial property. In the following discussion we will also use codimension to mean the difference of the dimension of a pair of sets.

For simplicity, we consider only the case of knots. Observe that in the vector space of ordered quadruples of points in  $\mathbb{R}^3$ , the subset for which the four points are collinear has algebraic codimension 4. Given four points  $a, b, c$ , and  $d$  on the unit circle, the space of knots  $K$  of degree  $d$  (for  $d \geq 2$ ) projects onto the space of quadruples of points in  $\mathbb{R}^3$ . Therefore the set of knots  $K$  of degree  $d$  for which  $K(a), K(b), K(c)$ , and  $K(d)$  are collinear has codimension 4 as well, as does the analogous set in the space of quintuples  $(K, a, b, c, d)$ , where  $a, b, c$ , and  $d$  are four distinct points on the circle. By projection, the set of pairs  $(K, a)$  for which there exists  $b, c$ , and  $d$  such that  $K(a), K(b), K(c)$ , and  $K(d)$  are collinear has codimension at least 1. Except for an algebraic subset of the set of knots, the set of  $a$  for a knot  $K$  for which  $b, c$ , and  $d$  can be found with this property is polynomial, i.e. finite. Such  $b, c$ , and  $d$  would have to exist in order for  $\pi_L(\cdot, a)$  to be three-to-one. Thus, part III of the lemma is proved for knots.

The rest of the lemma can be proved in the same fashion, namely by keeping track of the codimension of certain sets. Informally, a set of algebraic codimension  $n$  can be called an  $n$ -fold coincidence. Parts I and II of the lemma hold because, given points  $a$  and  $b$  on a link  $L$ , it would take a 2-fold coincidence for the tangent to  $L$  at  $b$  to contain  $a$ , and allowing a

to vary, it would take a 1-fold coincidence in the choice of  $a$ , or allowing  $a$  to vary, a 1-fold coincidence for the choice of  $b$ . Part IV of the lemma holds because, given  $a$ ,  $b$ , and  $c$  on a link  $L$ , it would take a 3-fold coincidence for  $a$ ,  $b$ , and  $c$  to be collinear and for the tangent lines at  $b$  and  $c$  to be coplanar.

For part V, the case when condition IV of the lemma fails typifies the method of proof. Informally, at a special point  $a$  for which  $\pi_L(\cdot, a)$  is somewhere three-to-one, three arms of the projection of the link meet at a point and it would take a coincidence for there to be a fourth arm at the point or for two of the arms to have the same slope. Near  $a$  the front two arms cross at a point and it would take a coincidence for that crossing to travel parallel to the third arm instead of passing through it.

Geometrically, it would take a 6-fold coincidence for five given points on a link  $L$  to be collinear, and it would take a 5-fold coincidence for four given points on  $L$  to be collinear and for two of the tangent lines to be coplanar. So in either case it would take a 1-fold coincidence in the choice of  $L$  for such a set of points to exist. Finally, consider collinear four points  $a$ ,  $b$ ,  $c$ , and  $d$  on  $L$  and let  $l_a$ ,  $l_b$ ,  $l_c$ , and  $l_d$  be the tangent lines at these points. The set of lines that intersect  $l_a$ ,  $l_b$ , and  $l_c$  sweeps out a surface, and it is a 1-fold coincidence in the choice of  $l_d$  for it to be tangent to that surface. If it is not tangent, then  $\pi_L(\cdot, a)$  will look as it does in case IV of Figure 1.  $\square$

### 3. KNOTS IN GENERAL POSITION

The arguments in this section follow that of [6] and [5]. The only new feature is the notion of topological non-trivial quadriseccants, which we will need to generalize the main theorem to arbitrary knots.

We begin with a simple lemma and a definition:

**Lemma 3.1.** *Let  $C$  be a compact set in  $\mathbb{R}^n$ . Then not every point of  $C$  lies between two other points of  $C$ .*

*Proof.* If  $p$  is any point in  $\mathbb{R}^n$ , then a point  $q \in C$  which is farthest from  $p$  has this property, because if  $q$  lay between two other points, one of them would be still farther away.  $\square$

**Definition 3.2.** *A secant of a link  $L$  with no extra interior intersections with  $L$  is topologically trivial if its endpoints lie on the same component of  $L$ , and if it, together with one of the two arcs of this component, bounds a disk whose interior does not intersect  $L$ . The disk may intersect itself and the secant. A quadriseccant  $\overline{ad}$  with middle points  $b$  and  $c$  is topologically trivial if any of the secants  $\overline{ab}$ ,  $\overline{bc}$ , and  $\overline{cd}$  are. Similarly for a triseccant.*

**Lemma 3.3.** *A knot in general position has a topologically non-trivial quadriseccant.*

*Proof.* Let  $K$  be a polynomial knot in general position. Let  $M$  be the set of unordered pairs of points of  $S^1$ , or equivalently the set of secants of  $K$ .  $M$  is topologically a Möbius strip. We define  $O$  to be the subset  $M$  consisting of those pairs of

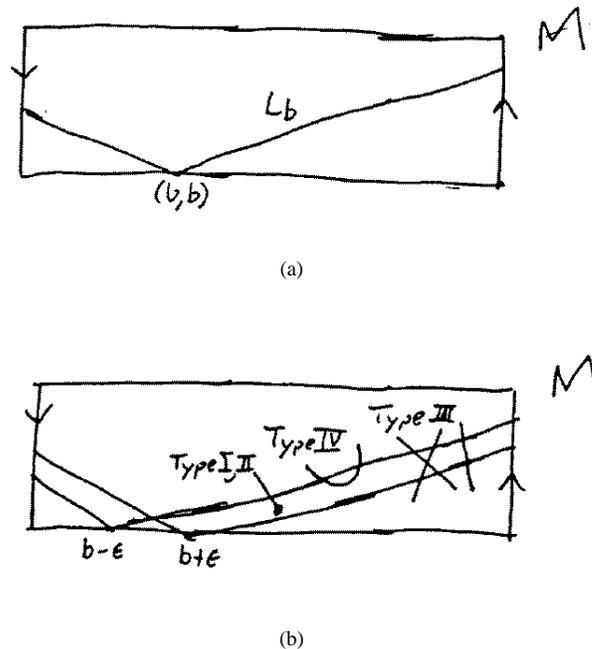


Figure 2

points  $(a, b)$  with the property that at least one point of  $K$  lies between  $K(a)$  and  $K(b)$ . Lemma 2.4 has implications about the local structure of  $O$ . For fixed  $a$ , the set  $L_a$  of all  $(a, b)$  in  $M$  is a line segment which wraps around  $M$  as in Figure 2(a). The intersection  $O \cap L_b$  is a finite set. If  $b$  is a generic point, the topology of  $\pi_K(\cdot, a)$ , and therefore the topology of  $O \cap L_a$ , cannot change as we vary  $a$  slightly. But if  $a$  is a special point, the topology of  $O \cap L_a$  changes as illustrated in Figure 2(b). For example, if  $a$  is a special point at which condition IV of Lemma 2.4 for  $\pi_K(\cdot, a)$  fails, then there exist three points  $b$ ,  $c$ , and  $d$  so that  $a, b, c$ , and  $d$ , in that order, make a quadriseccant of  $K$ . The triseccants  $a, c, d$  and  $a, b, d$  represent the same point of  $O$ , and if condition V of Lemma 2.4 holds, they represent arms of  $O$  that cross. Meanwhile the triseccant  $a, b, c$  represents a point of  $O$  that lies elsewhere along  $L_a$ .

It follows that  $O$  is the image of a self-transverse smooth immersion of a 1-manifold, and a self-crossing corresponds to a quadriseccant. If  $C$  is a curve of points in  $O$  which does not “make turns” at the self-crossings, then  $C$  is a continuous curve of triseccants.

The significance of  $O$  is that it is an obstruction to the following construction: Recall that on a Möbius strip, there are two kinds of properly embedded arcs, non-separating arcs and separating arcs. Suppose that  $A$  is a non-separating arc of  $M$  which avoids  $O$ . Then  $A$  corresponds to a family of secants whose interiors do not intersect  $K$ . This family of secants induces a map  $D$  from the unit disk to  $\mathbb{R}^3$  whose boundary is  $K$  and whose interior does not intersect  $K$ . By Dehn’s lemma,  $K$  is trivial.

Suppose that  $K$  has no quadriseccants. Then  $O$  is an em-

bedded 1-manifold. By elementary homology theory, if  $O$  obstructs all non-separating arcs, there is a circular component  $C$  of  $O$  which winds around  $M$  either one or two times. The curve  $C$  is a continuous family of trisecants. We consider the corresponding families of points  $\{a, b\}_t$  and  $m_t$ , with  $t \in S^1$ , such that  $K(m_t)$  lies between  $K(a_t)$  and  $K(b_t)$ . If  $C$  winds once around  $M$ , the endpoints travel half way around  $S^1$  and then switch places, and since  $m_t$  is trapped between them, it must jump discontinuously, a contradiction. If  $C$  winds twice around  $M$ , the endpoints each wind once around  $S^1$ , and therefore so does  $m_t$ . Thus, every point of  $K$  lies between two other points, which contradicts Lemma 3.1.

Topological non-triviality is achieved by a modification of this construction. Let  $O'$  be the subset of  $O$  consisting of topologically non-trivial trisecants and quadrisecants which are non-trivial at the middle secant. Observe that  $O'$  is also the image of a smooth immersion: If  $O'$  contains a self-intersection point of  $O$  but does not contain all four arms of the self-intersection, then it must contain exactly two arms, and they must be opposite rather than adjacent. In this case the self-intersection point is a quadrisecant which is topologically trivial on one side. Therefore if  $O'$  has a self-crossing, it corresponds to a topologically non-trivial quadrisecant. If there are no such quadrisecants,  $O'$  must also have a circular component  $C$  with all of the properties mentioned above, provided that  $O'$  is also an obstruction to all non-separating arcs  $A$ .

Let  $A$  be a non-separating arc which avoids  $O'$ . We may choose  $A$  to be transverse to  $O$ . As before, we construct the disk  $D_A$  from the secants of  $A$ , but this time  $D_A$  does not avoid  $K$ . Consider a point where  $A$  intersects a topologically trivial trisecant  $T$ . By hypothesis there exists a disk  $D_T$  which bounds a secant of  $T$  and an arc of  $K$ . Using  $D_T$  and a tubular neighborhood of  $K$ , we may alter  $D_A$  to obtain a disk  $D'_A$  which avoids  $K$  in the vicinity of  $T$ , according to Figure 3(a). We may similarly modify  $D_A$  in the vicinity of a quadrisecant  $Q$  which is topologically trivial in the middle, as in Figure 3(b). In this fashion we obtain a disk  $D$  whose interior avoids  $K$  as before, and Dehn's lemma applies.  $\square$

#### 4. LINKS IN GENERAL POSITION

The result of this section is a completion of analogous results in [5] and [6]. The arguments there roughly correspond to the  $\omega_1 \neq 0$  case of the proof, although the argument in [6] is somewhat more general than this special case.

**Lemma 4.1.** *Every non-trivial link  $L$  in general position has a topologically non-trivial quadrisecant.*

*Proof.* We may assume without loss of generality that no component of  $L$  bounds a disk whose interior avoids  $L$ .

Let  $K$  be a component of  $L$ . Let  $M_K$  be the Möbius strip of secants of  $K$ , and let  $O'_K$  be the corresponding set of topologically non-trivial trisecants and quadrisecants which are non-trivial in the middle. As before,  $O'_K$  must be an obstruction to non-separating arcs  $A$ , and we obtain a circle  $C$  which winds around  $M$ . If the middle points of  $C$  also lie on  $K$ , we may

apply the proof of the previous lemma. But the middle points may lie on some other component  $H$  of  $L$ . In this case, the secants of  $C$  induce a map  $f$  from a surface  $E$  to  $\mathbb{R}^3$ , where  $E$  is either an annulus or a Möbius strip, depending on whether  $C$  winds once or twice around  $K$ . We may choose  $f$  so that the median of  $E$  maps to the middle points of the trisecants of  $C$ .

The set of lines  $l$  perpendicular to  $H$  at a given point  $p$  is homeomorphic to a circle, and the corresponding set  $T$  of all ordered pairs  $(l, p)$  is homeomorphic to a torus. We may orthogonally project each trisecant  $t \in C$  to a line perpendicular to  $H$ , i.e. a member of  $T$ , thereby obtaining a map  $f$  from  $C$  to  $T$ . Since  $C$  is a circle, this map has an ordered pair of winding numbers  $(\omega_1, \omega_2)$  which are well-defined up to an orientation of  $C$ . There are three cases to consider, depending on the values of the winding numbers.

Suppose that  $\omega_1 = \omega_2 = 0$ . We construct a disk whose boundary is  $K$  and whose interior avoids  $L$ . The map  $f$  intersects  $H$  at the median, but it may also intersect  $K$  at some other points, because  $C$  may include some quadrisecants which are topologically trivial on one side. In this case we can modify  $f$  according to the prescription in Figure 3(a) to obtain a map  $f'$  which avoids  $K$  in the interior and which agrees with  $f$  in a neighborhood of the median. Since both winding numbers are zero, we may now homotop  $f'$  in a neighborhood of  $H$  to obtain a map  $f''$  which avoids  $H$  and is constant on the median of  $E$ . Finally, we identify the median of  $E$  to a point to obtain a space  $E'$  and a map  $f'''$ . If  $E$  is a Möbius strip,  $E'$  is a disk, but if  $E$  is an annulus,  $E'$  is two disks identified at a point. Either way, we obtain the desired spanning disk, which we may convert to an embedded disk by Dehn's Lemma.

Suppose instead that  $\omega_2 = 0$  but  $\omega_1 \neq 0$ . Then we extend  $E$  to a line bundle  $E'$  and extend  $f$  linearly to a map  $f' : E' \rightarrow \mathbb{R}^3$ . We can homotop  $f'$  in a neighborhood of  $H$  without changing its values in  $E' \setminus E$  to a map  $f''$  which has constant value  $p$  on the zero section of  $E'$ , but we cannot make  $f''$  avoid  $H$ . Let  $p \in H$ . As before, we identify the zero section of  $E'$  to a point and obtain a space  $E''$ , and correspondingly alter  $f''$  to obtain a map  $f''' : E'' \rightarrow \mathbb{R}^3$ . This time the intersection number between  $H$  and  $f'''$  at  $p$  is  $\omega_1$ . But since  $f'''$  is a closed map from the pseudo-manifold  $E''$  to  $\mathbb{R}^3$ , it induces a well-defined homology class in the infinite homology of  $\mathbb{R}^3$ .  $H$  induces another such homology class, and by elementary homology theory, the total intersection number between  $f'''$  and  $H$  must be zero. The map  $f'''$  must intersect  $H$  at another point, and therefore  $f''$  does also. Suppose that  $f''(x)$  is this point, with  $x \in E'$ . The point  $x$  cannot be in  $E$ , therefore  $f'(x) = f''(x)$ . Since  $f'$  is linear on the fibers of  $E'$ , the image under  $f'$  of the fiber containing  $x$  yields a quadrisecant  $Q$ . The quadrisecant  $Q$  is necessarily topologically non-trivial, because if the intersection points of  $Q$  are labeled in order as  $a, b, c$ , and  $x$ , then  $b, x \in H$  and  $a, c \in K$ .

The only remaining possibility is that  $\omega_2 \neq 0$ . In this case, every point of  $H$  lies between two points of  $K$ . We may repeat the whole argument with each component of  $L$  playing the role of  $K$ , thereby obtaining a function  $f$  from components of  $L$  to components of  $L$  such that every point of  $f(K)$  lies between two points of  $K$ . The map  $f$  must have at least one circular orbit, and we may set  $C$  to be the set in  $\mathbb{R}^3$  which is the union

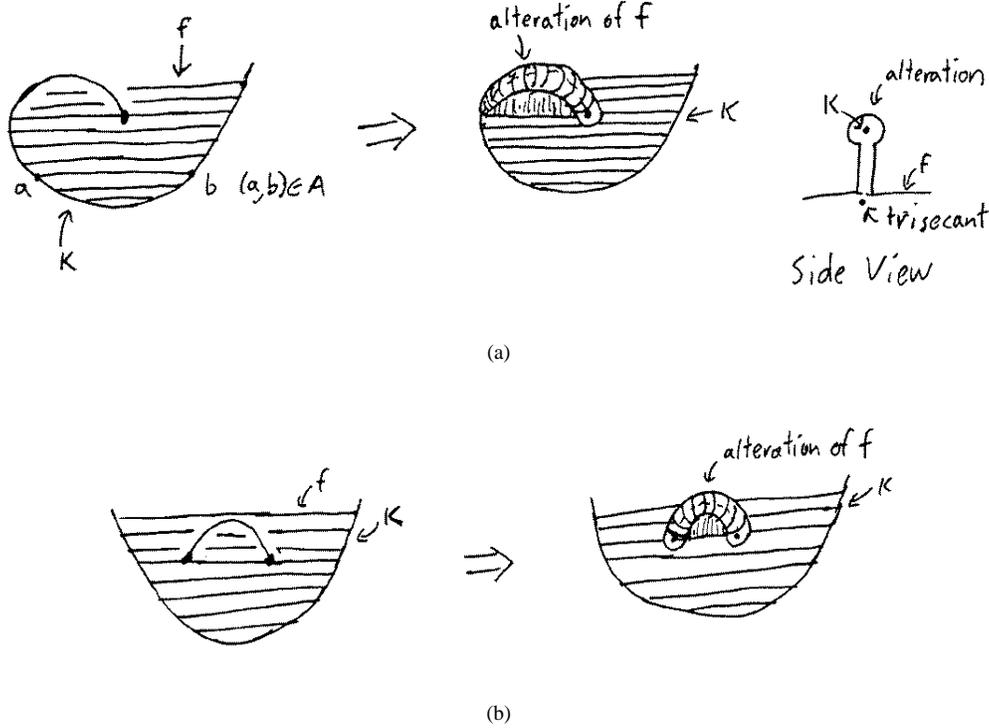


Figure 3

of all components of  $L$  in this orbit. Evidently,  $C$  is a compact set and every point of  $C$  lies between two other points of  $C$ , a contradiction by Lemma 3.1.  $\square$

## 5. ARBITRARY TAME KNOTS AND LINKS

**Definition 5.1.** A link  $L$  in  $\mathbb{R}^3$  is tame if there exists a homeomorphism  $h$  of  $\mathbb{R}^3$  which carries  $L$  to a polynomial link, or equivalently a piecewise linear or smooth link.

**Lemma 5.2.** If  $L$  is a tame link, there exists a homeomorphism  $h$  of  $\mathbb{R}^3$  which maps  $L$  to a smooth link with  $h$  smooth on  $\mathbb{R}^3 - L$ .

*Proof.* Let  $K$  be a tame knot and let  $h$  be an arbitrary homeomorphism such that  $h(K)$  is smooth. Using a tubular neighborhood of  $h(K)$ , we can choose  $T_i$ , with  $i \geq 1$ , to be a sequence of nested, parallel tori converging to  $h(K)$ . Let  $T'_i = h^{-1}(T_i)$ . By the theory of triangulations and smoothings of 3-manifolds (see [4, p. 217]), there exists a sequence of smooth tori  $T''_i$ , with each  $T''_i$  lying between  $T'_i$  and  $T'_{i+1}$ , and a sequence of diffeomorphisms  $h'_i: T''_i \rightarrow T'_i$  such that  $h'^{-1}_i$  and  $h^{-1}|_{T'_i}$  are isotopic as maps from  $T'_i$  to  $\mathbb{R}^3 - K$ . Furthermore, we can arrange that the distance between  $h'^{-1}_i$  and  $h^{-1}$  goes to zero as  $i \rightarrow \infty$ . By the isotopy condition, the  $h'_i$ 's may be extended smoothly to the each region between  $T'_i$  and  $T'_{i+1}$  and the region outside

$T_1$  to obtain a diffeomorphism  $h': \mathbb{R}^3 - K \rightarrow \mathbb{R}^3 - h(K)$ . Because of the distance condition, we can continuously extend  $h'$  to  $K$  by setting it equal to  $h$  on  $K$ . This continuous extension is the desired map.

The proof in the case of links is similar.  $\square$

We are now in a position to prove Theorem 1.3. In fact, we can prove something slightly stronger:

**Theorem 5.3.** If  $L$  is a non-trivial tame link in  $\mathbb{R}^3$ , then  $L$  has a quadrisecant, none of whose component secants are subsets of  $L$ .

*Proof.* Let  $L$  be a non-trivial, tame link and let  $h$  be a homeomorphism given by Lemma 5.2. Let  $N$  be a tubular neighborhood of  $h(L)$ , let  $N'$  be the normal bundle of  $h(L)$ , and choose a diffeomorphism  $n: N \rightarrow N'$ . Consider a sequence of links  $L_i$  such that  $h(L_i)$  is disjoint from  $L$  and  $n(h(L_i))$  is a smooth section. Choose the sequence so that  $h(L_i)$  converges smoothly to  $h(L)$ , i.e.  $n(h(L_i))$  converges smoothly to the zero section. Since  $h$  is a diffeomorphism outside of  $L$ , we may choose each  $L_i$  to be a polynomial link in general position.

By hypothesis, each  $L_i$  has the same isotopy type as  $L$ , and in particular each  $L_i$  is non-trivial. Therefore each  $L_i$  has a topologically non-trivial quadrisecant  $Q_i$ . By compactness,  $\{Q_i\}$  has a convergent subsequence in the space of line segments in  $\mathbb{R}^3$ ; we may suppose without loss of generality that the original sequence converges. The resulting limit is a se-

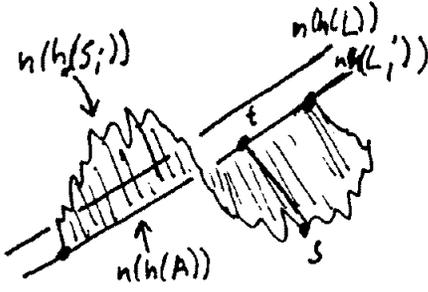


Figure 4

cant of  $L$ . We must show that the endpoints and middle points of the quadriseccants do not converge together.

For each  $i$ , let  $S_i$  be a topologically non-trivial secant of  $L_i$  and suppose that the  $S_i$ 's converge to a point  $p$  on  $L$ . Let  $B$  be a round, open ball in  $N'$  centered at  $n(h(p))$ . Then there exists an  $i$  such that  $S_i$  and an arc  $A$  of  $L_i$  with the same endpoints as  $S_i$  are both contained in  $h^{-1}(n^{-1}(B))$ . For each point  $s \in n(h(S_i))$ , we consider the line segment from  $s$  to  $t$ , where  $t$  is the point in  $n(h(L_i))$  which lies in the same fiber of  $N'$  as  $s$ , as illustrated in Figure 4. Since  $n(h(L_i))$  is a section,  $t$  is unique. The union of these line segments is the image of a spanning disk of  $n(h(A \cup S_i))$  which does not intersect  $n(h(S_i))$ . Therefore  $S_i$  is topologically trivial, a contradiction.

The proof that the limit of the  $S_i$ 's is not a subset of  $L$  is similar.  $\square$

Corollary 1.4 follows from this theorem:

*Proof.* Let  $\{T_i\}$  be a non-trivially linked collection of solid tori. For each  $i$  and each  $n > 0$ , let  $l_{i,n}$  be the shortest non-contractible loop in  $T_i$  which is homotopically  $n$  times the core of  $T_i$ . Let  $l_i$  be a shortest member of the set  $\{l_{i,n}\}$ . If we let  $D$  and  $D'$  be two disjoint, non-separating disks in  $T_i$  for some  $i$ , then we see that the length of  $l_{i,n}$  is bounded below by  $n$  times the distance between  $D$  and  $D'$ . Therefore  $l_i$  exists, although it may not be unique.

Suppose that for some  $a, b \in S^1$ ,  $l_i(a) = l_i(b)$ . Then we can divide  $l_i$  into two loops from  $l_i(a)$  to itself. At least one of these loops must be non-contractible and both loops are shorter, which is a contradiction. Thus, each  $l_i$  is an embedding. If we let  $L$  be the union of the images of the  $l_i$ 's, then  $L$  is a satellite link of the  $T_i$ 's. By a theorem in knot theory [8, p. 113],  $L$  must be a non-trivial link if the  $T_i$ 's are.

Since a geodesic in a smooth manifold with smooth boundary must be  $C^1$  (see [1]; a proof was also suggested to the author by Tom Ilmanen),  $L$  must be a tame link. By the preceding theorem,  $L$  must have a quadriseccant  $Q$  such that no component secant of  $Q$  is contained in  $L$ . Suppose that a component secant  $S$  of  $Q$  were contained entirely inside some  $T_i$ . Let  $p$  be a path which goes from one endpoint of  $S$  to the other. Then we can divide  $l_i$  into two paths  $q_1$  and  $q_2$  to make two loops  $q_1p$  and  $q_2p$  whose composition is homotopic to  $l_i$ .

At least one of these loops must be non-contractible, therefore they cannot both be shorter. Therefore each component secant of  $Q$  must have one point which lies outside the  $T_i$ 's.

Finally, suppose that  $P(x, y, z)$  is a non-trivial polynomial whose zero set contains  $\partial T_i$  for all  $i$ . Then the restriction of  $P$  to the line containing  $Q$  must be non-trivial and must have at least 8 real roots, counting multiplicity. Therefore  $P$  has degree at least 8.  $\square$

The author once believed that the loop in a solid torus which is the shortest non-zero multiple of the core is necessarily homotopic to the core. However, this is false by an example of Doug Jungreis. We can consider the region  $S$  in  $\mathbb{R}^3$  which consists of the set of points  $(x, y, z)$  such that:

$$|x - \sin(L_1^2 y)/L_1 - \sin(L_2^2 z)/L_2| < \varepsilon,$$

where  $L_1$  is very large,  $L_2$  is much larger still, and  $\varepsilon$  is much smaller than  $1/L_2$ . The region  $S$  could be described as a corrugated sheet, and it has the property that if  $a, b \in S$  and the straight-line distance from  $a$  to  $b$  is greater than 1, then this distance is much less than the length of the shortest path in  $S$  from  $a$  to  $b$ . If  $M$  is a smooth Möbius strip in  $\mathbb{R}^3$  whose tangent plane varies slowly, we can approximate  $M$  with a solid torus  $T$  which is topologically a tubular neighborhood of  $M$  but which is geometrically quite different:  $T$  is the union of a thick tube centered around the boundary of  $M$  and a corrugated sheet which approximates the interior of  $M$ , as shown in Figure 5. Clearly the shortest non-trivial loop in  $T$  stays close to the boundary of  $M$  and is therefore homotopically twice the core.

It is easy to show that the bound in Corollary 1.4 is the best possible: If we choose two numbers  $r_1 > r_2$ , then the surface given by:

$$(x^2 + y^2 + z^2 - r_1^2 - r_2^2)^2 - 4(x^2 + y^2)r_1^2 = 0$$

is a torus. If  $r_1 > 2r_2$ , we can multiply two such surfaces together to obtain two linked tori.

## 6. QUESTIONS OPEN TO THE AUTHOR

The most serious shortcoming of Corollary 1.4 is the fact that it only applies to closed surfaces in  $\mathbb{R}^3$ , while the usual context for studying degrees of real algebraic surfaces is  $\mathbb{R}P^3$ . We may view a subset of  $\mathbb{R}^3$  as a subset of  $\mathbb{R}P^3$  which is disjoint from the "plane at infinity", which is a copy of  $\mathbb{R}P^2$ . We may define a *flat* plane in  $\mathbb{R}P^3$  to be the image of the plane at infinity under a projective transformation of  $\mathbb{R}P^3$ , and a *topological* plane to be the image of the plane at infinity under a homeomorphism of  $\mathbb{R}P^3$ . This brings us to the following generalization of the results of this paper:

**Conjecture 6.1.** *If a non-trivial link in  $\mathbb{R}P^3$  is disjoint from some topological plane, then it has four collinear points.*

**Conjecture 6.2.** *If an algebraic surface in  $\mathbb{R}P^3$  is disjoint from some topological plane and bounds a collection of non-trivially linked solid tori, then the surface has degree at least 8.*

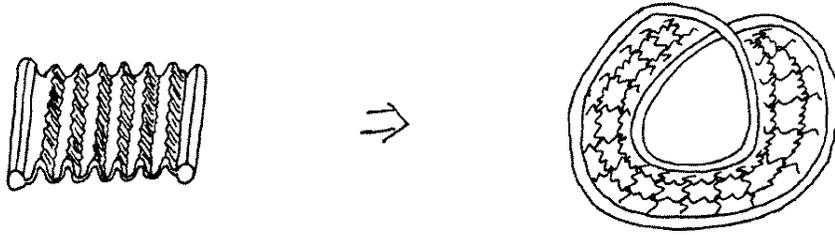


Figure 5

The following questions have also eluded the author:

**Conjecture 6.3.** *If an algebraic surface in  $\mathbb{R}^3$  is a smooth torus which is knotted on the outside, then it has degree at least eight.*

**Conjecture 6.4.** *Every wild arc in  $\mathbb{R}^3$  has infinitely many*

*quadriseccants.*

**Question 6.5.** *What is the lowest possible degree of a polynomial surface in  $\mathbb{R}^3$  which is the boundary of the tubular neighborhood of a trefoil knot?*

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