2020-10-27 Kähler geometry

M Fano manifold

$$\omega = i g_{i\bar{j}} dz^i \wedge d\bar{z}^j \in 2\pi c_1(M)$$

$$\mathrm{Ric}(\omega) - \omega = i \partial\bar{\partial} F_\omega \qquad F = F_\omega$$ omitted.

$$F \in C^\infty(M)$$

Aubin's continuity method

$$*_t \qquad \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{-t\varphi + F}$$

sol for $t = 1 \Longrightarrow$ KE metric

$$S = \{ t \in [0,1] \mid *_t \text{ has a solution} \}.$$

$$\left\{ \begin{array}{ll} \cdot\ S \text{ is non-empty} & \rightsquigarrow 0 \in S, \text{ Yau's solution} \\ \cdot\ S \text{ is open} & \rightsquigarrow \text{implicit fn thm (Delanoë)} \\ \cdot\ S \text{ is closed} & \rightsquigarrow \boxed{\text{difficult.}} \end{array} \right.$$

$$\Longrightarrow S = [0,1] \Longrightarrow S \ni 1$$

Closedness

When $\underset{\in S}{t_i} \to t_\infty$, $\underline{t_\infty \in S\ !}$

$\mathrm{Ric}(g_{t_i}) > t\, g_i$

$\ast_t \Leftrightarrow \mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega_0 > t\omega_t$

Cheeger-Colding theory

$(M, g_{t_i}) \longrightarrow M_\infty$ Gromov-Hausdorff

Demerit.
- It is difficult to understand the singularity
- It is unclear how to use K-stability.

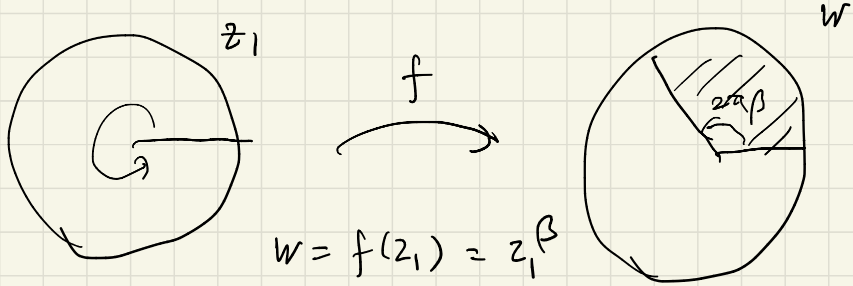
Another continuity method using cone angle KE metrics. $2\pi\beta \leftarrow$ cone angle

M (compact) Kähler manifold

D smooth divisor $D = \{z_1 = 0\}$ locally

local coord $(z_1, \cdots, z_m)$

local model of a Kähler metric with cone angle $2\pi\beta$



$$w = f(z_1) = z_1^{\beta}$$

$$f^*\left(i\,dw\wedge d\bar{w} + i\,dz_2\wedge d\bar{z}_2 + \cdots + i\,dz_m\wedge d\bar{z}_m\right)$$

$$= \beta^2 |z_1|^{2(\beta-1)}\, i\,dz_1\wedge d\bar{z}_1 + i\,dz_2\wedge d\bar{z}_2 + \cdots + i\,dz_m\wedge d\bar{z}_m$$

<u>Def</u> A Kähler form ω with cone angle $2\pi\beta$ along a divisor D is when ω is uniformly equivalent to

$$\omega_{cone} = \beta^2 \frac{i\,dz_1\wedge d\bar{z}_1}{|z_1|^{2-2\beta}} + \sum_{j=2}^{m} i\,dz_j\wedge d\bar{z}_j$$

where $D\cap U = \{z_1 = 0\}$,

i.e. $\exists\, c_1, c_2 > 0$ constant s.t.

$$c_1\,\omega_{cone} \le \omega \le c_2\,\omega_{cone}.$$

<u>Rm</u> $\beta = 1 \Longrightarrow \omega$ is smooth.

Def Ricci current $\mathcal{R}ic(\omega)$

$$\mathcal{R}ic(\omega) = -i\partial\bar\partial \log \omega^m$$ distribution derivative

Lemma Write $Ric(\omega)$ to be the Ricci form on $M \setminus D$. Then

$$\mathcal{R}ic(\omega) = Ric(\omega) + 2\pi(1-\beta)[D].$$

as a current.

☺ Poincaré-Lelong equation

$$\frac{i}{2\pi}\partial\bar\partial \log |z_1|^2 = [D]$$ ☺

$\boxed{\text{Next we assume } M \text{ Fano.}}$

Suppose, $[D] = \lambda c_1(M)$

D smooth

(In algebraic geometry it is unknown whether we can take $\lambda = 1$)

Def A Kähler form ω in $2\pi c_1(M)$ with cone angle $2\pi\beta$ along D is a Kähler-Einstein metric if

$$\mathcal{R}ic(\omega) = \mu\omega + 2\pi(1-\beta)[D] \quad *_\beta$$

(A version of twisted KE metric)

Remark $\mu = 1-(1-\beta)\lambda.$

☺ $\mathrm{Ric}(\omega) = \mu\omega + 2\pi(1-\beta)[D].$

$$\underbrace{\mathrm{Ric}(\omega)}_{2\pi c_1(M)} = \underbrace{\mu\omega}_{2\pi\left(\mu c_1(M) + (1-\beta)\lambda c_1(M)\right)} + 2\pi(1-\beta)[D]$$

$\therefore\ 1 = \mu + (1-\beta)\lambda.$ ☺

$E = \{\beta \in [0,1] \mid *_\beta \text{ has a solution}\}.$

(i) $E \neq \emptyset$ Brendle, Jeffres-Mazzeo-Rubinstein

(ii) E is open CDS $\rightsquigarrow$ can be reduced orbifold

(iii) E is closed if M is K-polystable. Donaldson + Song . Wang

ω_0 smooth Kähler form $[\omega_0] = 2\pi c_1(M).$

$\mathcal{H} = \{\varphi \in C^\infty(M) \mid \omega_0 + i\partial\bar\partial\varphi > 0\}.$

$\beta \in [0,1]$

$\hat{\mathcal{H}}_\beta = \{\varphi \in C^\infty(M\setminus D) \mid \omega_0 + i\partial\bar\partial\varphi$ Kähler form of cone angle $2\pi\beta$ along $D\}.$

$K_M^{-\lambda}$

Example $\varepsilon > 0$

$\varphi \in \mathcal{H} \Rightarrow \varphi + \varepsilon|s|_h^{2\beta} \in \hat{\mathcal{H}}_\beta$

where $s \in H^0(M,[D])$, $(s) = D$, h metric of $[D]$

Lemma For $\varphi_\beta \in \hat{\mathcal{H}}_\beta$ we put $\omega_\beta = \omega_0 + i\partial\bar\partial\varphi_\beta$.

If we define h_β by

$$Ric(\omega_\beta) = \mu\,\omega_\beta + (1-\beta)\,2\pi[D] + \sqrt{-1}\,\partial\bar\partial h_\beta.$$

then

$$h_\beta = h_{\omega_0} - (1-\beta)\log\|S\|^2_{h_0^\lambda} - \log\frac{\omega_\beta^m}{\omega_0^m} - \mu\varphi_\beta$$

h_0 metric K_M^{-1} $\rightsquigarrow$ h_0^λ metric for $K_M^{-\lambda} = [D]$.

where $c_1(h_0) = \omega_0$, $Ric(\omega_0) - \omega_0 = i\partial\bar\partial h_{\omega_0}$

$\because$ $i\partial\bar\partial(\mathrm{RHS})$

$$= \cancel{Ric(\omega_0)} - \omega_0 + (1-\beta)\lambda\,\omega_0 - (1-\beta)[D]$$

$$+ Ric(\omega_\beta) - \cancel{Ric(\omega_0)} - \mu(\omega_\beta - \omega_0)$$

$$-1 + (1-\beta)\lambda + \mu = 0$$

$$= Ric(\omega_\beta) - (1-\beta)[D] - \mu\,\omega_\beta$$

$$= \sqrt{-1}\,\partial\bar\partial h_\beta.$$

$\therefore$

Cor $\varphi_\beta \in \hat{\mathcal{H}}_\beta$ is a solution to $(*_\beta)$

$$\Longleftrightarrow \quad \omega_{\varphi_\beta}^m = e^{-\mu\varphi_\beta + h_{\omega_0}}\,\frac{\omega_0^m}{\|S\|_h^{2(1-\beta)}} \qquad (*_\beta)$$

Test config $\mathcal{M} \longrightarrow \mathbb{C}$ can be extended to ⑦

$(\mathcal{M}, \mathcal{D}) \longrightarrow \mathbb{C}$ naturally $\qquad M \supset D$

(M_0, D_0) the central fiber.

<u>Def - Lemma</u> (Chi Li).

Let X be a holomorphic vector field on M_0 such that $\exp(tX)$ leaves D_0 invariant

$$\left(\begin{array}{l} X \text{ is tangent to } D_0 \\ X = Y z_1 \frac{\partial}{\partial z_1} + \sum X^j \frac{\partial}{\partial z_j} \\ \qquad (Y, X^j \text{ holomorphic}) \end{array} \right)$$

$$i(X)\omega = -\bar{\partial} u_X$$

$$f_\beta(X) = \int_{M_0} u_X (\mathrm{Ric}(\omega) - \omega) \wedge \omega^{n-1}$$

$$+ (1-\beta)\left(\lambda \int_{M_0} u_X \omega^n - 2\pi \int_{D_0} u_X \omega^{n-1}\right)$$

"log Futaki invariant"

$$-1 + (1-\beta)\lambda = -\mu$$

(1) $f_\beta(X)$ is indep of $\varphi_\beta \in \hat{\mathcal{H}}_\beta$.

(2) If $\exists$ K-E with cone angle $2\pi\beta$

then $f_\beta = 0$.

$(\mathcal{M}, \mathcal{D})$

log DF is defined as follows.

$$\frac{w_k}{k d_k} = F_0 + F_1 k^{-1} + \cdots \longrightarrow \begin{cases} F_j(M_0) \\ F_j(D_0) \end{cases}$$

$$\log DF_\beta(\mathcal{M}, \mathcal{D}) =: f_\beta(M_0, D_0)$$

$$= -F_1(M_0) - (1-\beta)\frac{m\lambda}{2}\left(F_0(M_0) - F_0(D_0)\right)$$

(Chi Li).

Proof of closedness of E

Step 1 (Sun) For any test configuration

$f_0(M_0, D_0) \geq 0$.

[That is, for cone angle 0, always K-semi stable.]

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$\boxed{\text{Step 2}}$ $\beta_i \in E$, $\beta_i \longrightarrow \beta_\infty$. Suppose $\beta_\infty \notin E$.

- Gromov-Hausdorff limit

$$(M, \omega_{\beta_i}, D) \longrightarrow (M_\infty, D_\infty)$$

is in fact an algebraic limit.

$M_\infty \neq M$.

- $\exists$ test configuration $(\mathcal{X}, \mathcal{D}) \to \mathbb{C}$ with central fiber (M_∞, D_∞).

(Cheeger-Colding-Tian cone angle version)

- (M_∞, D_∞) has a weak solution of KE cone angle β_∞, so that

$$f_{\beta_\infty}(M_\infty, D_\infty) = 0.$$

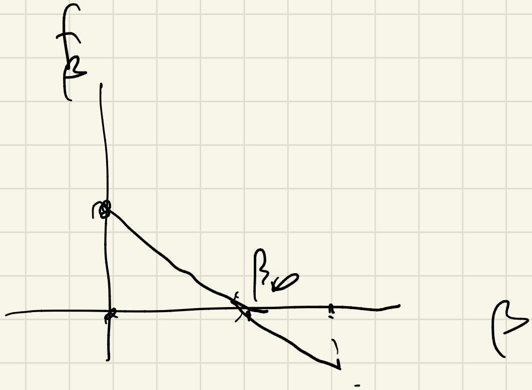
Step 3 As a function β

- $f_\beta(M_\infty, D_\infty) = f(M_\infty) + (1-\beta)\frac{m\lambda}{2}\left(F_0(M_\infty) - F_0(D_\infty)\right)$

: linear function in β.

- $f_0(M_\infty, D_\infty) \geq 0$, $f_{\beta_\infty}(M_\infty, D_\infty) = 0$

Step 1 (Sun)



$$\therefore\ f_1(M_\infty, D_\infty) \le 0.$$

$$\text{"}$$

$$0 \le \ -F_1(M_\infty)$$

$\uparrow$ K-polystability.

$$DF = -F_1(M_\infty) = 0$$

$M \neq M_\infty$ not product configuration

$\Longrightarrow$ contradiction.