2021-11-01 Kähler geometry

Let $\Lambda^* = \Lambda -$ zero section.

locally on U $\quad \Lambda^*|_U \cong U \times \mathbb{C}^*$

e frame

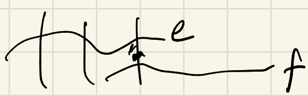
$$ze(p) \longleftrightarrow (p, z)$$

$z \neq 0$

$$\tilde{\theta} := \frac{dz}{z} + \theta$$

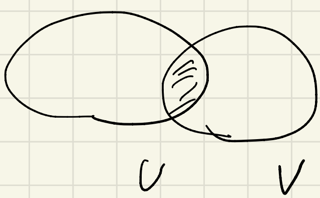
Proposition: $\tilde{\theta}$ defines a global 1-form on Λ^*.

(This is the connection form for the principal $\mathbb{C}^*$-bundle Λ^*. $\frac{dz}{z}$ is the Maurer-Cartan form.)



frame fields. f a frame on V

$f = e\sigma$ on $U \cap V$



$\sigma \in C^\infty(U\cap V, \mathbb{C}^*)$

$$p = \overset{ez}{z e} = \overset{fw}{w f} = e\sigma w$$

$\therefore z = \sigma w$

$$\nabla e = e\theta \text{ on } U, \quad \nabla f = f\theta' \text{ on } V.$$

$$\nabla e \cdot \sigma + e \cdot d\sigma = e\theta\sigma + e\, d\sigma = e\sigma\cdot\theta'$$

$$\theta\sigma + d\sigma = \sigma\theta'$$

$$\theta' = \theta + \sigma^{-1} d\sigma.$$

$$\frac{dw}{w} + \theta' = \frac{d(\sigma^{-1} z)}{\sigma^{-1} z} + \theta + \sigma^{-1} d\sigma$$

$$= \frac{dz}{z} - \cancel{\frac{\sigma^{-1} d\sigma\, \sigma^{-1}}{\sigma^{-1}}} + \theta + \cancel{\sigma^{-1} d\sigma}$$

$$= \frac{dz}{z} + \theta.$$

It is generally true that for e.g. vector bundle, connection gives a global 1-form on its associated principal bundle, P_{GL}

P_{GL} = the set of all frames of E_p for all $p \in N$.

= "frame bundle"

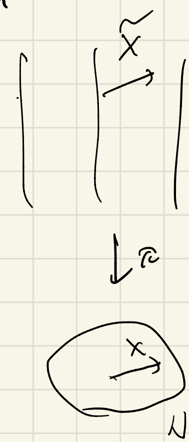
" Kobayashi - Nomizu : Theory of connections ".

Then

$$d\tilde{\theta} = d(d \log z + \theta) = \overset{\pi^*}{d\theta} = 2\pi\,\Omega$$

$$\Lambda^* \xrightarrow{\pi} N$$

Suppose the action of K lifts to Λ.
Then $X \in k$ defines a vector field $\tilde{X}$ on Λ^* with $\pi_*(\tilde{X}) = X$.



$$i(\tilde{X})\, d\tilde{\theta} = i(\tilde{X})\, \pi^* d\theta$$
$$= i(\pi_* \tilde{X})\, d\theta$$
$$= i(X)\, d\theta = i(X)\, 2\pi\Omega$$

Suppose also $\tilde{\theta}$ is K-invariant.

(This can be done if K is compact.)

$L_X \tilde{\theta} = 0$.

$\therefore\ d\, i(\tilde{X})\tilde{\theta} + i(\tilde{X})\, d\tilde{\theta} = 0$,

$\therefore\ d(\tilde{\theta}(\tilde{X})) = -i(X)\, 2\pi\Omega$

Conclusion $\underline{\tilde{\theta}(\tilde{X})}$ is a Hamiltonian function for $\underline{X \in k}$ (Homework.).

descends to N.

GL(n, $\mathbb{R}$)

GL(n, $\mathbb{C}$).

Prop.

$$\begin{aligned} \mu : M &\longrightarrow \mathfrak{k}^* \\ p &\longmapsto \mu(p) \end{aligned}$$

$$\langle \mu(p), X \rangle = \hat{\theta}(\tilde{X})(p)$$

is a moment map.

(Homework : Prove the equivariance.)

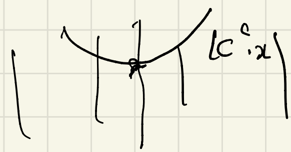
$\Lambda \to N$ ample line bundle over a compact complex mfd N. $\to$ alg.

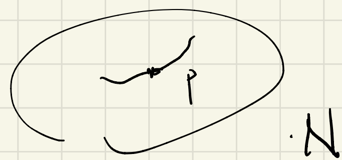
Assume K^c action on N lifts to Λ.

then also to $\Lambda^{-1} \leftarrow \mathcal{O}_Z(-1)$. $\pi : \Lambda^{-1} \to N$

$p \in N$ is polystable $\overset{\text{def}}{\Longleftrightarrow}$ for $x \in \pi^{-1}(p)$, $x \neq 0$, $K^c \cdot x$ is closed.

($K^c \cdot p$ is polystable)





Kempf-Hess thm

$p \in N$ is polystable $\Longleftrightarrow$ $K^c \cdot p$ has a zero of the moment map $\mu : N \to k^*$.

(Fujiki-Donaldson picture. applies Kempf-Hess to the infinite dimensional moment map geometry.)

Proof of Kempf-Ness

General fact (1) Hilbert-Mumford criterion.

Stability holds for K^c if and only if stability holds for every one-parameter subgroup of K^c (i.e. $\mathbb{C}^*$-action).

General fact (2)

Let h be a Hermitian metric on Λ.

Then there is a unique connection satisfying

~~$\nabla_\xi T - \nabla_T \xi = [\xi, T]$~~

(i) $\nabla h = 0$

(ii) connection form is Type $(1,0)$.

(Chern connection).

Concretely, the connection θ and curvature Θ are given by

local frame e on U, $h(e,e) =: h_U$

$$\theta = h^{-1}\partial h = \partial \log h.$$

$\partial \log h_U$

$$\Theta = \bar\partial\theta = \bar\partial\partial \log h = -\partial\bar\partial \log h.$$

(7)

$$\left(c_1(\Lambda) := \left[\frac{i}{2\pi} \Theta\right] = \left[-\frac{i}{2\pi}\partial\bar\partial \log h_U\right] \right)$$

homework
show this class is indep of h.)

Λ ample $\Leftrightarrow$ $-\partial\bar\partial \log h_U$ is positive.

$\Leftrightarrow$ $\partial\bar\partial h_U^{-1}$ is positive (Λ^{-1})

Based on (1), (2), let Γ be an orbit of the $\mathbb{C}^*$-action on Λ^{-1}.

choose a trivialization of Λ^{-1} as $U \times \mathbb{C}$.
Then Γ is of the form.

$$\Gamma|_U = \Gamma \cap (U \times \mathbb{C}) = \{ \underbrace{g(t)\,(p, w)}_{(p(t),\, w(t))} \in U \times \mathbb{C} \}$$

($g(t) \in \mathbb{C}^*$)

$$\left.\frac{d}{dt}\right|_{t=0} g(t)\,(p, w) = \underbrace{X_p}_{N} + \underbrace{f(p)\, w \frac{\partial}{\partial w}}_{\text{fiber.}}$$

$$=: \widetilde{X}_p \quad \text{on } \Lambda^{-1}$$

Let $\partial \gamma \in \Lambda^{-1}$ (8)

$H : \Gamma \longrightarrow \mathbb{R}$ h_0^{-1} $w \in \mathbb{C}^*$

be the function defined by

$$H(\gamma) = \log |\gamma|^2 = \log (h^{-1} w \bar{w}).$$

Then

$$\partial H = - h^{-1} \partial h + \frac{dw}{w}$$

$$\left(w \in \Lambda^{-1}, \quad z = \frac{1}{w} \in \Lambda \right) \quad \text{on } \Lambda^*$$

$$\rightsquigarrow \partial H = - h^{-1} \partial h - \frac{dz}{z} = - \tilde{\theta} \quad \text{on } (\Lambda^{-1})^*$$

we studied last time

$$u_X = \tilde{\theta}(\tilde{X}) = - X H$$

So, H has a critical point along Γ if and only if $u_X = 0$ somewhere on Γ.

Moreover H is a convex function

$$i \partial \bar{\partial} H = i \partial \bar{\partial} \log |\gamma|^2 = - i \partial \bar{\partial} \log h$$

$$= i H > 0. \quad \text{since } \Lambda \text{ ample.}$$

Thus Λ_x has a zero along Γ (9)

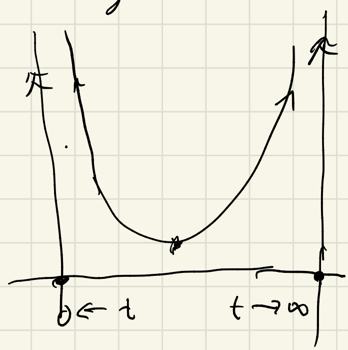
$\Longleftrightarrow$ moment map has a zero along Γ

$\Downarrow$

The convex function H has a critical point

$\Longleftrightarrow$ Γ is closed in Λ^{-1} - zero section.

$\Longleftrightarrow$ the orbit is closed.



☺ QED.

<u>Summary</u>

H proper $\Longleftrightarrow$ the orbit is polystable in Λ^{-1}

H bounded from below $\Longleftrightarrow$ semistable

H not bounded from below $\Longleftrightarrow$ unstable.

Hermitian Yang-Mills (Kobayashi - Hitchin)
$\to$ H is Donaldson function.

cscK $\to$ H is K-energy.