(1)

X : real vector field $\longleftrightarrow$ $\frac{1}{2}(X - iJX)$

If ig corresponds to holo vector field, then it corresponds to a killing vector.

This means

$$(ig)^i \frac{\partial}{\partial z^i} = \frac{1}{2}(X - iJX) \quad \underline{\text{holo}}$$

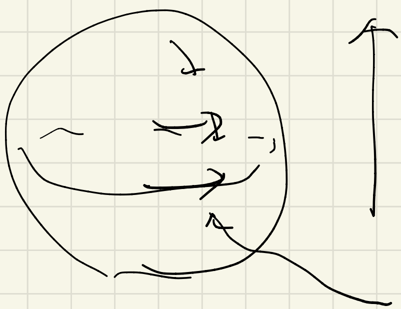
$$\rightsquigarrow X \text{ is killing}$$

$$g^i \frac{\partial}{\partial z^i} = \frac{1}{2}(Y - iJY) \quad \text{holo}$$

$$\longrightarrow JY \text{ is killing}$$

$$\text{\rotatebox{90}{$=$}}$$

$$J\left(\frac{\partial g}{\partial x^i} \frac{\partial}{\partial x^i}\right) = J \operatorname{grad} g$$



g

$\nexists$ grad g

J grad g

Last time we showed

$$\exists \text{ cscK} \Longrightarrow \mathfrak{h}(M) \text{ reductive.}$$

Example

$E \subset \widehat{\mathbb{CP}^2}$ = the blow-up of $\mathbb{CP}^2$ at a point.

$$\begin{tikzcd} E \arrow[d] & \widehat{\mathbb{CP}^2} \arrow[d, "\pi"] \\ pt & \mathbb{CP}^2 \end{tikzcd}$$

$$\begin{array}{ccc} \mathrm{Aut}(\widehat{\mathbb{CP}^2}) & \longrightarrow & \mathrm{Aut}(\mathbb{CP}^2) \\ \cup & & \\ \sigma & \longmapsto & \left(\begin{array}{c|cc} * & * & * \\ \hline 0 & * & * \\ 0 & * & * \end{array}\right) \end{array}$$

$$\sigma(E) = E$$

$$\pi(E) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

not reductive

$\mathfrak{a}$ + ss.

Any Kähler class on $\widehat{\mathbb{CP}^2}$ cannot have a cscK metric.

$c_1(\widehat{\mathbb{CP}^2}) > 0$ but $\nexists$ KE.

$$\mathrm{Aut}(\widehat{\mathbb{CP}^2}) = \left\{ \begin{matrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{matrix} \right\} \quad \text{not reductive}$$

$$\mathrm{Aut}(\hat{\hat{\hat{\mathbb{CP}}}}{}^2) = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \quad \text{reductive.}$$

Siu, Nadel, Tian $\exists$ KE.

Def $\mathfrak{g} := \{ u \in C^\infty(M) \otimes \mathbb{C} \mid Du = 0$

$\int_M u \, \omega^n = 0 \}$

$= \{ u \in C^\infty(M) \otimes \mathbb{C} \mid \nabla^i u \frac{\partial}{\partial z^i} \in \mathfrak{h}(M),$

$\int_M u \, \omega^n = 0 \}$

Lie alg in terms of Poisson bracket

$\{u, v\} = u^i v_i - v^i u_i \in \mathfrak{g}$

$\in \mathfrak{g}$

$$= g^{i\bar{j}} \left(\frac{\partial u}{\partial \bar{z}^j} \frac{\partial v}{\partial z^i} - \frac{\partial v}{\partial \bar{z}^j} \frac{\partial u}{\partial z^i} \right)$$

Rem If $H^1(M) \neq 0$ then

$\mathfrak{h}(M) = \mathfrak{a} + \mathfrak{g}$ as a Lie alg

↑ abelian

If $H^1(M) = 0$ then $\mathfrak{h}(M) = \mathfrak{g}$.

④

Rem If $\mathrm{Lie}(G) = \hat{\mathfrak{g}}$, $G \subset \mathrm{Aut}(M)$, and if L is a holomorphic line bdle s.t. $c_1(L) = [\omega]$ ("L is ample") then G-action on M lifts L.

$$\begin{array}{c} \curvearrowright L \\ \downarrow \\ G \curvearrowright M \end{array}$$

Donaldson-Kronheimer (4-mfd).

Theorem (F)

$$f : \mathfrak{h}(M) \longrightarrow \mathbb{C}$$
$$\quad X \longmapsto -\int_M u_X \, S \, \omega^m$$

scal curv. (S)

$\left(\mathrm{grad}_g u_X = X, \ \int_M u_X \, \omega^m = 0 \right)$

is independent of $\omega \in [\omega_0]$.

In particular if $f \not\equiv 0$ then $\nexists$ cscK metric.

proof of last statement

If $\exists$ cscK metric so $S = \text{const}.$

$$f(x) = -\int u_X S \,\omega^n = -\underset{\text{const}}{S} \int_M u_X \,\omega^n$$

$$= 0.$$

☺

Lemma If $\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + t\varphi_{i\bar{j}} \quad + t \in [0, \varepsilon)$

and X is holo vector field then u and $\tilde{u}$

satisfying $\mathrm{grad}'_g u = X$, $\mathrm{grad}'_{\tilde{g}} \tilde{u} = X$

are related by

$$\tilde{u} = u + t u^i \varphi_i .$$

$$g^{i\bar{j}} \frac{\partial \varphi}{\partial \bar{z}^j} \frac{\partial \varphi}{\partial z^i}$$

☺ $\tilde{g}^{i\bar{j}} (u + t u^k \varphi_k)_{\bar{j}} = \tilde{g}^{i\bar{j}} (u_{\bar{j}} + t u^k \varphi_{k\bar{j}})$

$$= \tilde{g}^{i\bar{j}} \Big(u^k \underbrace{(g_{k\bar{j}} + t\varphi_{k\bar{j}})}_{\tilde{g}_{k\bar{j}}} \Big)$$

$$= u^i = X^i$$

$$\frac{d}{dt}\int (u + t u^k \varphi_k)\, \omega_t^m \Big|_{t=0} \quad \text{for convenience}$$

exercise.

$$= \int u^k \varphi_k\, \omega^m + \int u\, \Delta\varphi\, \omega^m$$

$$= -\int u\, \Delta\varphi \cdot \omega^m + \int u\, \Delta\varphi\, \omega^m = 0.$$

So if $\int_M u\, \omega^m = 0$ then $\int_M \tilde{u}\, \tilde{\omega}^m = 0$

☺

<u>Proof</u> of Thm (first part). just for convenience.

$$\frac{d}{dt}\int (u + t u^k \varphi_k)\, S_t\, \omega_t^m \Big|_{t=0}$$

$$= \int u^k \varphi_k \cdot S\, \omega^m - \int u\,(\Delta^2 \varphi + R^{i\bar{j}} \varphi_{i\bar{j}})\,\omega^m$$

$$+ \int u S\, \Delta\varphi\, \omega^m \qquad g + t\nabla\bar{\nabla}\varphi$$

☺ $$S_t = - g_t^{i\bar{j}} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det g_t$$

$$\frac{dS_t}{dt} = -\Delta^2 \varphi - R^{i\bar{j}} \varphi_{i\bar{j}}$$

☺

$$= -\int u\, \nabla^k S \cdot \nabla_k \varphi\, \omega^m - \int u\, \Delta^2 \varphi \cdot \omega^m$$

$$+ \int \Big(\nabla_{\bar j} u\, R^{i\bar j} + u\, \underbrace{R^{i\bar j}{}_{,\bar j}}_{S^{,i}} \Big) \varphi_i\, \omega^m$$

$$= \int \big(\nabla_k u \cdot \cancel{\nabla^k} S + u \cancel{\Delta S} \big) \varphi\, \omega^m - \int \Delta^2 u \cdot \varphi\, \omega^m$$

$$- \int \big(u_{,i\bar j} R^{i\bar j} + \nabla_{\bar j} u \cdot R^{i\bar j}{}_{,i} + \cancel{u_{,i} S^{i}}$$

$$+ \cancel{u\, S^{i}{}_{,i}} \big) \varphi\, \omega^m$$

$$= -\int \big(\Delta^2 u + u_{,i\bar j} R^{i\bar j} + \nabla_{\bar j} u \cdot S^{\bar j} \big) \varphi\, \omega^m$$

$$= -\int Du \cdot \varphi\, \omega^m = 0$$

$$\Big(\because\ \underline{\text{grad}' u \ \text{ holo}} \Big).$$

$$\left(\begin{aligned} &\underline{\nabla_{\bar j} \nabla^i u = 0} \iff \nabla_{\bar j} \nabla_{\bar i} u = 0 \\ &\iff Du = \nabla^{\bar j} \nabla^{\bar i}\ \nabla_{\bar j} \nabla_{\bar i} u. \end{aligned} \right)$$

☺

Rem $\underline{\underline{\mathrm{Fut}}} = f : \mathfrak{h}(M) \to \mathbb{C}$

Futaki invariant.

Now we turn to K-stability.

Application of an idea in GIT.

What is geometric invariant theory.

- To construct a good moduli space of algebraic geometric objects.
  (such as holo vector bundles or varieties.)
- You often have to take a quotient by a complex Lie group action.
- If you take a quotient naively, you do not get a good space.
  e.g. can not be Hausdorff.
  can not be compactified.
- If you discard "unstable" orbits you can get a good moduli space.

due to Mumford. ("L'Enseignement")

<u>Book.</u> Mumford – Fogarty – Kirwan.

Stability in algebraic sense.

(later symplectic sense in terms of moment map <u>Donaldson–Kronheimer</u>)

Let V be a vector space over $\mathbb{C}$.

$G \subset SL(V)$

<u>Def</u> $p \in V$ $(p \neq 0)$ is stable

$\iff$ (1) the orbit $G \cdot p$ is closed.

(2) the stabilizer

$$\mathrm{Stab}(p) = \{ g \in G \mid gp = p \}$$

is finite subgroup.

<u>Def</u> $p \in V$, $p \neq 0$, is polystable

$\iff$ only (1) holds.

Def $p \in V$ is semi-stable

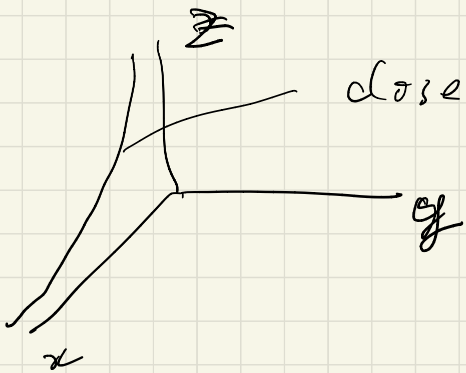
$\iff$ the closure of $G \cdot p$ does not contain $0 \in V$.

(polystable, stable $\Rightarrow$ semi stable.)

Example. $V = \mathbb{C}^3$

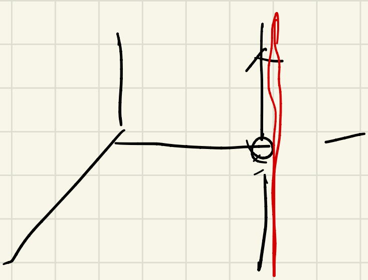
$G = \mathbb{C}^*$, $t(x, y, z) = (t^{-1}x, y, tz)$

(a) If $x \neq 0$, $z \neq 0$



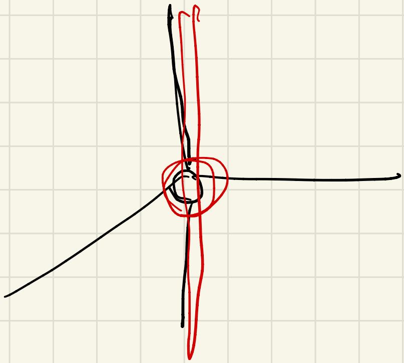
closed orbit

(b) $t(0, 1, z) = (0, 1, tz)$



closure $\not\ni 0$.

(c) $t(0,0,z) = (0,0,tz)$



unstable.

(d) $t(0,1,0) = (0,1,0)$

$\mathrm{Stab}(0,1,0) = \mathbb{C}^*$

polystable.

$V \ni p \neq 0$

$\mathbb{P}(V)$ = projective space

= $\{$lines in V, through $0\} \ni [p]$

$Z \subset \mathbb{P}(V)$ submfd.

$$\mathcal{O}_{\mathbb{P}(V)}(-1) = \{([p], x) \in \mathbb{P}(V) \times V \mid x \in [p]\} \longrightarrow \mathbb{P}(V)$$

tautological line bdle (Milnor-Stasheff)

$$c_1(\mathcal{O}(-1)) = -\frac{i}{2\pi}\partial\bar{\partial}\log(1+|t^1|^2+\cdots+|t^m|^2)$$

on $\{z^0 \neq 0\}$, $t^i = \frac{z^i}{z^0}$

$\mathcal{O}(-1)$ - zero section $\cong V-\{0\}$

$$\mathcal{O}_Z(-1) = \mathcal{O}(-1)\Big|_Z$$

Suppose $G \subset SL(V)$ preserves $\underline{\underline{Z}}$.

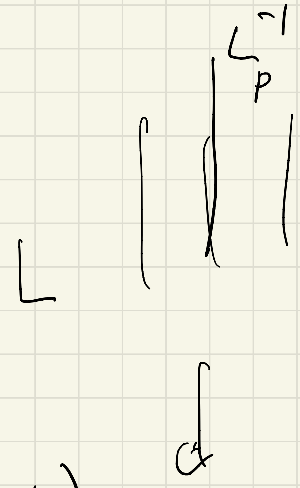
Let (Z, ω) compact Kähler manifold.

$[\omega] \in H^2(M:\mathbb{Z})$ / torsion.

$L \to Z$ holomorphic line bundle with $c_1(L) = [\omega]$.

(such an L is called an ample line bdle.)

Suppose a reductive Lie group G acts on L as bundle isomorphisms.



<u>Def</u> $p \in Z$ (semi) stable

$\underset{\text{def}}{\iff}$ $\forall x \neq 0$ in L_p^{-1}

$G \cdot x$ closed

and finite stabilizer

(resp. $\overline{Gx} \cap$ zero section $= \emptyset$)

$\mathbb{P}(V) \supset Z \ni p$

This def is motivated by Kodaira embedding thm.

Let $L \to M$ be an ample line bdl.

$\exists\, \nu$ s.t. $\forall k \geq \nu$, $s_0, \ldots, s_{N_k} \in H^0(M, L^k)$ basis

$$\Phi_k : M \longrightarrow \mathbb{P}^{N_k}(\mathbb{C})$$

$$p \longmapsto (s_0(p), \ldots, s_{N_k}(p))$$

gives an embedding. $\left(\mathbb{P}^{N_k}(\mathbb{C}) \cong \mathbb{P}(H^0(M, L^k)^*)\right)$