2020 · 10 · 14

Lemma.

$$\phi[\omega] = \frac{1}{Z}\, x^{o(\omega)} \sqrt{2}^{\,l(\omega)}$$

$$x = \frac{p}{\sqrt{2}(1-p)}.$$ $l(\omega) := \#$ loops in loop representation.

pf: $$\phi[\omega] = \frac{1}{Z}\left(\frac{p}{1-p}\right)^{o(\omega)} 2^{k(\omega)}$$

$$l(\omega) = k(\omega) + k(\omega^*) - 1$$

$$\#V - o(\omega) + f(\omega) = 1 + k(\omega)$$

$$\Updownarrow$$

$$k(\omega^*)$$

$$k(\omega^*) = k(\omega) + o(\omega) + \text{const}.$$

$$l(\omega) = 2k(\omega) + o(\omega) + \text{const}.$$

$$\phi[\omega] = \frac{1}{Z}\left(\frac{p}{1-p}\right)^{o(\omega)} \sqrt{2}^{\,l(\omega) - o(\omega)}$$

$$= \frac{1}{Z}\left(\frac{p}{\sqrt{2}(1-p)}\right)^{o(\omega)} \sqrt{2}^{\,l(\omega)}.$$

$$p = \frac{\sqrt{2}}{1+\sqrt{2}}, \quad x = \frac{\frac{\sqrt{2}}{1+\sqrt{2}}}{\sqrt{2}\,\frac{1}{1+\sqrt{2}}} = 1.$$

FK-Ising critical

$$\phi[\omega] = \frac{1}{Z}\sqrt{2}^{\,\ell(\omega)}.$$

Def. edge fermionic observable

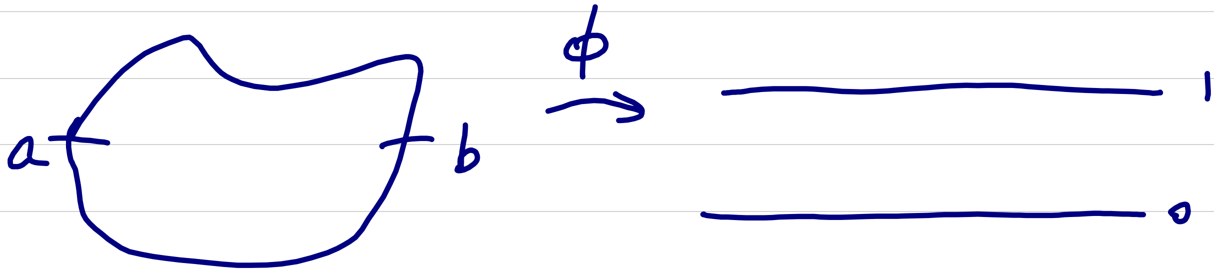
$$F_\delta(e) = \mathbb{E}\left[\mathbb{1}_{\{e\in\gamma\}} \exp\left(\frac{i}{2} W_\gamma(e,b)\right)\right]$$

γ: the exploration path from a_δ to b_δ in the loop representation

$W_\gamma(e,b)$: the winding between e and b_δ.

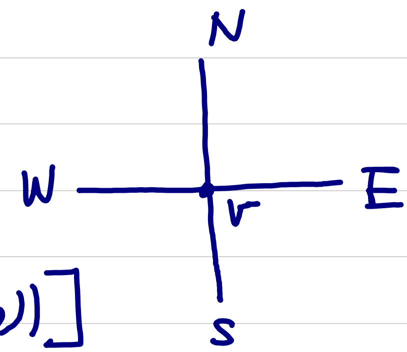
vertex fermionic observable

$$F_\delta(v) = \frac{1}{2}\sum_{e\sim v} F_\delta(e).$$



Lemma. edge fermionic observable:

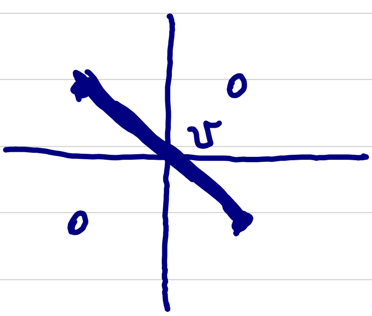
$$F(N) - F(S) = i\,(F(E) - F(W)).$$



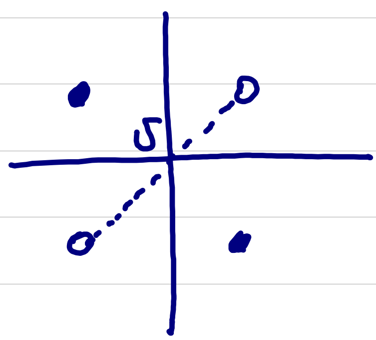
Pf: $$F(e) = \mathbb{E}\left[\mathbb{1}\{e\in\gamma\}\exp\left(\tfrac{i}{2}W_\gamma(e,b)\right)\right]$$

$$= \frac{1}{Z}\sum_{\omega} e_\omega$$

$$e_\omega = \sqrt{2}^{\,\ell(\omega)}\,\mathbb{1}\{e\in\gamma\}\exp\left(\tfrac{i}{2}W_\gamma(e,b)\right).$$



ω



$s(\omega)$

$$F(e) = \sum e_\omega = \frac{1}{2}\sum\left(e_\omega + e_{s(\omega)}\right)$$

$$N_\omega + N_{s(\omega)} - S_\omega - S_{s(\omega)}$$

$$= i\left(E_\omega + E_{s(\omega)} - W_\omega - W_{s(\omega)}\right) \quad (\star)$$

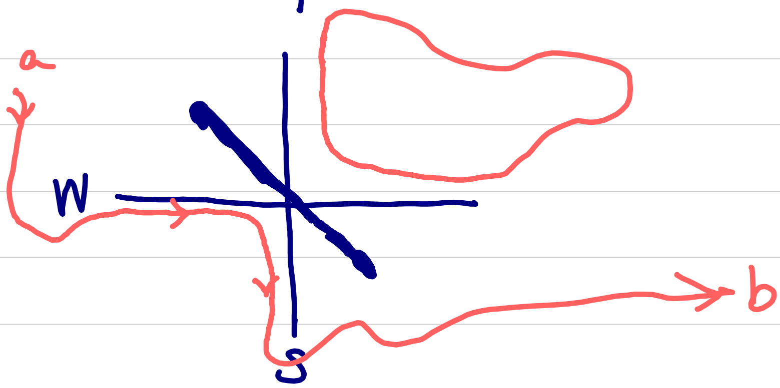
$$N_\omega = \sqrt{2}^{\,\ell(\omega)} \underline{\mathbb{1}\{N \in \gamma\}} \exp\left(\tfrac{i}{2} W_\gamma(N, b)\right)$$

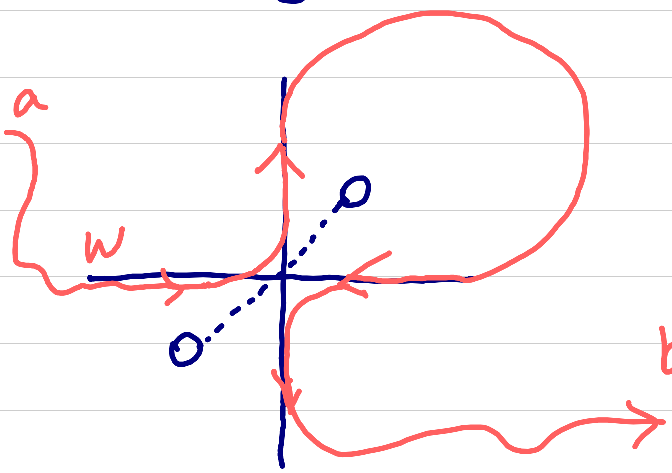
case 1. $\gamma(\omega)$ does not go through any of N, E, S, W.

(★) holds.

case 2. $\gamma(\omega)$ goes through two edges around v.

we may assume it enters through W, exits through S.





config.	W	E	N	S
ω	W_ω	0	0	$e^{\frac{\pi i}{4}} W_W$
$s(\omega)$	$\frac{1}{\sqrt{2}} W_W$	$\frac{1}{\sqrt{2}} W_W e^{\frac{\pi i}{2}}$	$\frac{1}{\sqrt{2}} W_W e^{-\frac{\pi i}{4}}$	$\frac{1}{\sqrt{2}} W_W e^{\frac{\pi i}{4}}$

$$S_\omega = \sqrt{2}^{\,\ell(\omega)} \mathbb{1}\{s \in \gamma\} \exp\left(\frac{i}{2} \underline{W_\gamma(s,b)}\right)$$

$$W(w, b) = W(s, b) - \frac{\pi}{2}$$

$$W(S, b) = W(W, b) + \frac{\pi}{2}$$

$$S_\omega = \sqrt{2}^{\,\ell(\omega)} \mathbb{1}\{s \in \gamma\} \exp\left(\frac{i}{2} W_\gamma(W,b)\right) e^{\frac{\pi i}{4}}$$

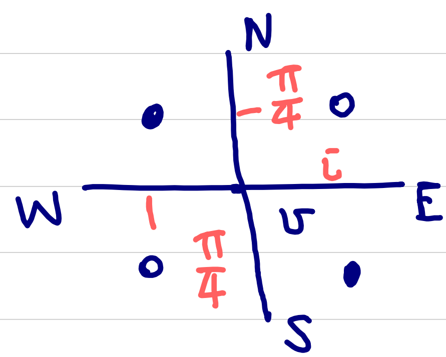
$$W_{s(\omega)} = \frac{1}{\sqrt{2}} W_\omega$$

$$\begin{aligned} W(W,b) &= W(N,b) + \frac{\pi}{2} \\ &= W(E, b) - \pi \\ &= W(S, b) - \frac{\pi}{2} \end{aligned}$$

(A) holds.

lemma. the vertex fermionic observable is s-holo.

Pf. $F_\delta(v) = \frac{1}{2} \sum F_\delta(e)$



$F_\delta(v) = \frac{1}{2}\left(F(N) + F(E) + F(S) + F(W)\right).$

assume $b_\delta \longrightarrow$

$F(e) = \mathbb{E}\left[\mathbb{1}_{\{e \in \gamma\}} \exp\left(\frac{i}{2} W_\gamma(e, b)\right)\right]$

$// \ \ell(e).$

$F(N) = (1 - i)\, a_N$, $a_N \in \mathbb{R}$

$F(E) = i\, a_E$, $a_E \in \mathbb{R}$

$F(S) = (1+i)\, a_S$, $a_S \in \mathbb{R}$

$F(W) = a_W$, $a_W \in \mathbb{R}$

$F(N) - F(S) = i\,(F(W) - F(W)).$

$a_E = a_S - a_N, \quad a_W = a_N + a_S.$

$F(v) = F(N) + F(S) = F(W) + F(E)$

$$F(V) = F(N) + F(S)$$

$\ell(N)$ $\quad$ $\ell(S)$

$$F(N) = P_{\ell(N)}[F(V)]$$

$$F(S) = P_{\ell(S)}[F(V)].$$

$$F(\ell) = P_{\ell(e)}[F(V)].$$

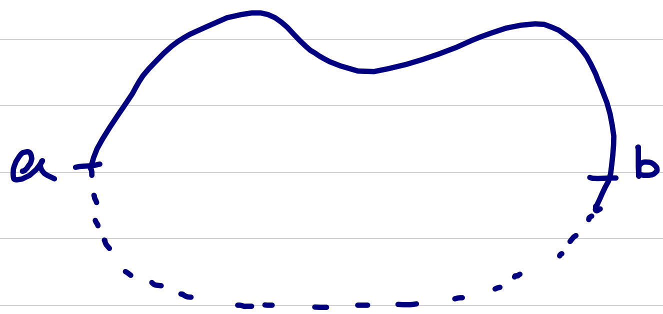
the vertex fermionic observable is s-hol.

$$H_\delta(A) = 1.$$

$$H_\delta(B) - H_\delta(W) = |P_{\ell(e)}[F_\delta(x)]|^2$$

$H_\delta^\bullet$ subharmonic

H_δ° superharmonic.



$H_\delta^\bullet = 1$ on (ba)

$H_\delta^\bullet \to 0$ on (ab)

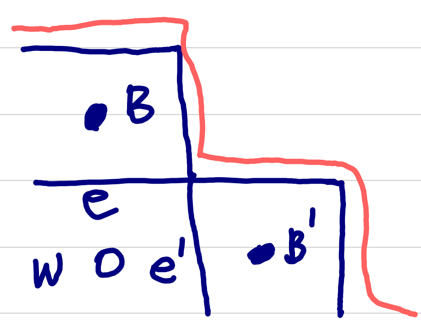
uniformly away from a, b

$H_\delta^\circ = 0$ on $(a^* b^*)$

$H_\delta^\circ \to 1$ on $(b\, a)$

uniformly away from a, b.

Pf: $H_\delta^\bullet$ is constant along (ba)



$$H_\delta^\bullet(B) - H_\delta^\bullet(B')$$

$$= H(B) - H(W) - \left(H(B') - H(W)\right)$$

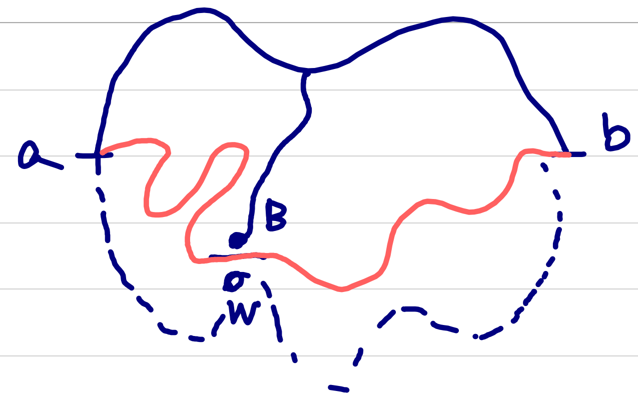
$$= |F(e)|^2 - |F(e')|^2 = 0$$

$$H^\bullet(A) = 1.$$

$H_\delta^\bullet = 1$ along $(b\,a)$

$H^\circ = 0$ along $(a^* b^*)$.

consider $H_\delta^\bullet$ along (ab)



$H^\bullet(B) - H^\circ(w) = |F(e)|^2$

$H^\circ(w) = 0$

$H^\bullet(B) = |F(e)|^2$

$= \phi[e \in \gamma]^2$

$\leq \phi[B \leftrightarrow (ba)]^2$

B is at distance r from (ba),

$U_\delta = B + [-r, r]^2$

$\phi[B \leftrightarrow (ba)]$

$\lesssim \phi^1_{U_\delta}[B \leftrightarrow \partial U_\delta]$

$H^\bullet(B) \lesssim \phi^1_{U_\delta}[B \leftrightarrow \partial U_\delta]^2 \to 0$ as $\delta \to 0$.

$H_\delta^\bullet \to 0$ along (ab)

uniformly away from a, b.

$H_\delta^\circ \to 1$ along (ba). uniformly away from a, b.

Lemma. $H_\delta \to \operatorname{Im}\phi$ locally uniformly

Pf: $h^\bullet_\delta$ preharmonic, same boundary data as $H^\bullet_\delta$.

h°_δ preharmonic, same boundary data as H°_δ.

$H^\bullet$ subharmonic, $H^\bullet_\delta \le h^\bullet_\delta$

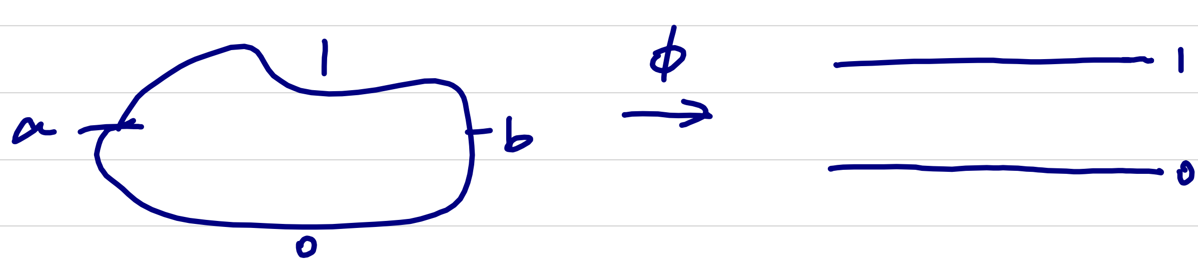
H° superharmonic, $H^\circ_\delta \ge h^\circ_\delta$.

$b_\delta \in \Omega_\delta$, $w_\delta \in \Omega^*_\delta$ neighbors

$$h^\circ_\delta(w) \le H^\circ_\delta(w) \le H^\bullet_\delta(b) \le h^\bullet_\delta(b)$$

Applying "convergence of Dirichlet boundary value solution" to h°_δ and $h^\bullet_\delta$

$h^\circ_\delta, h^\bullet_\delta \to \operatorname{Im}\phi$



$H_\delta \to \operatorname{Im}\phi$.

Thm. $\frac{1}{\sqrt{2\delta}} F_\delta \to \sqrt{\phi'}$. locally uniformly.

Pf. $\left(\frac{1}{\sqrt{2\delta}} F_\delta\right)_{\delta>0}$ tight $\nwarrow$

$\left(\frac{1}{\sqrt{2\delta_n}} F_{\delta_n}\right) \to f$. convergent subseq.

as F_{δ_n} is preholo, the limit f is holo.

$$H_{\delta_n}(y) - H_{\delta_n}(x) = \frac{1}{\delta_n} \operatorname{Im} \int_x^y F_{\delta_n}(z)^2 dz$$

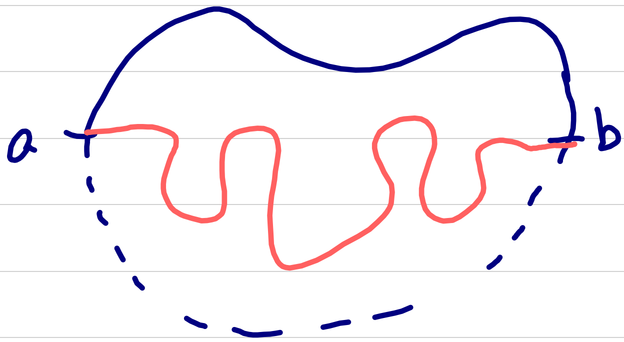
$$\downarrow \qquad\qquad \downarrow$$

$$\operatorname{Im}(\phi(y) - \phi(x)) \qquad\qquad \operatorname{Im} \int_x^y f(z)^2 dz$$

$$\underbrace{\operatorname{Im}(\phi(y) - \phi(x))}_{\text{holo.}} = \underbrace{\operatorname{Im} \int_x^y f(z)^2 dz}_{\text{holo}}.$$

$$\phi(y) - \phi(x) = \int_x^y f(z)^2 dz + \text{const}.$$

$$\phi'(y) = f(y)^2. \qquad f = \sqrt{\phi'}.$$



$\to \mathrm{SLE}\left(\frac{16}{3}\right)$