Monge-Ampere equation and complete Calabi-Yau metrics

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(joint work with Tristan Collins and Shing-Tung Yau)

Calabi-Yau Theorem

Let (X, J, ω) be a *compact* Kähler manifold with trivial canonical bundle. Then we have a following fundamental theorem of Yau regarding the existence of Ricci-flat metrics on X.

Theorem (Yau, 1978)

Let $[\omega] \in H^{1,1}(X, \mathbb{R})$ be a Kähler class of X. Then there exist a unique Ricci-flat Kähler metric $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ in $[\omega]$. Moreover, $\tilde{\omega}$ satisfies the complex Monge-Ampere equation

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Question

How does one construct non-compact (complete) Calabi-Yau metrics?

Complete Calabi-Yau metrics

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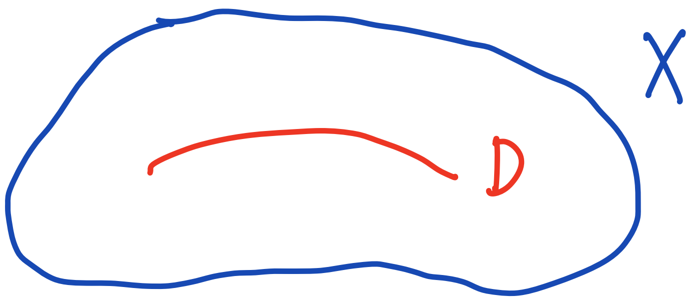
The focus of the rest of this talk will be on Step 1, as the second part is much better understood.

Quasiprojective varieties

The general set-up is the following: Take a compact Kähler manifold X and an anticanonical (or a fraction of) divisor $D \subset X$, then $M = X \setminus D$ is a non-compact Calabi-Yau manifold with infinite volume, and one tries to solve the complex Monge-Ampere equation on M.



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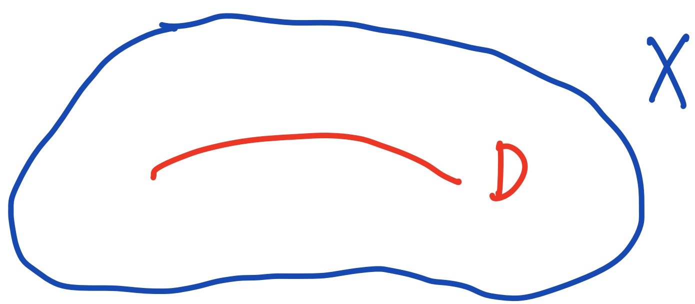

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Theorem (Tian-Yau)

Let X be a Fano manifold, and D be a smooth *anti-canonical divisor. Then M admits a complete Calabi-Yau metric $\tilde{\omega}$ which solves*

$$\tilde{\omega}^n = c\Omega \wedge \overline{\Omega}.$$

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4. The complex Monge-Ampere equation reduces to a simple ODE for φ, which has an explicit solution
$$\varphi(t) = \frac{n}{n+1} t^{\frac{n+1}{n}}.$$

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What if D is no longer smooth, but is simple normal crossing?

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Theorem (Collins-Li, 2022)

If $2 = k < n$, then generalized Tian-Yau metrics exist.

Main input: construction of a new model metric.

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5. The complex Monge-Ampere equation reduces to a Monge-Ampere equation for φ, which we would like to solve on $\mathbb{R}^k_+$

$$\left(\sum_{i=1}^{k} \frac{\partial\varphi}{\partial x_i}\right)^{n-k} \det D^2\varphi = const.$$

Homogenous solutions

If one further look for **homogenous** solutions

$$\varphi(t_1, \dots, t_k) = \left(\left(\sum_{i=1}^{k} t_i \right) v \left(\frac{t_1}{\sum_{i=1}^k t_i}, \dots, \frac{t_k}{\sum_{i=1}^k t_i} \right) \right)^{\frac{n+k}{n}},$$

then we reduce to a PDE for v on a simplex $P \subset \mathbb{R}^{k-1}$

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3. Critical points must be obtained using a min-max procedure.

The extension problem and boundary behaviour

The boundary behaviour is related to the extension of the ω_{model} to the non-generic region. The behaviour of the solution near the l dimensional boundary pieces should inform us of how to extend ω_{model} to the l-fold intersections of the D_i.

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3. In the normal directions, the leading sub-linear behaviour should be given by a homogenous solution of the generalized Calabi ansatz in lower dimension.

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$$v(x) = linear + \sum_{i,j=2}^{k-1} P_{ij} x'_i x'_j + c(x'_1)^{\frac{1}{n-k+1}} + l.o.t \tag{5}$$

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The proof relies largely on the results of Savin-Jhaveri, on the regularity of optimal transport problems with degenerate density.

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Conjecture (Liouville Theorem)

Let $\varphi : (\mathbb{R}_+)^l \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ *be a solution to the Monge-Ampere equation*

$$\begin{cases} \left(\sum_{i=1}^{l} \frac{\partial \varphi}{\partial x_i}\right)^k \det D^2\varphi = 1 \text{ in } (\mathbb{R}_+)^l \times \mathbb{R}^m \\ \sum_{i=1}^{l} \frac{\partial \varphi}{\partial x_i} = 0 \text{ on } \partial(\mathbb{R}_+)^l \times \mathbb{R}^m. \end{cases} \tag{6}$$

Then φ *is of the form*

$$\varphi(x) = c + \sum_{i=l+1}^{l+m} v_i x_i + \sum_{i,j=l+1}^{l+m} P_{ij} x_i x_j + p\varphi_{l,k}(x_1, \ldots, x_l)$$

Thank you!

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Happy anniversary YMSC!

Happy birthday Prof. Yau!