# Coherent-constructible correspondence

On the other hand, there exist results relating $\mathrm{Coh}(X)$ for toric varieties to $\mathrm{Con}(\mathbb{T})$ where $\mathbb{T}$ is a certain real torus obtained from the toric data of $X$. Combining this with the above theorem, we obtain statements of homological mirror symmetry. We first construct $\mathbb{T}$ and give examples.

Recall that $X = X_\Sigma$ is determined by a fan $\Sigma$ in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, for $N \cong \mathbb{Z}^d$ a lattice.

M is the dual lattice under a pairing which we write as $\langle m, n \rangle$ for $m \in M$, $n \in N$.

Writing $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, we let

$$\boxed{\mathbb{T} = M_{\mathbb{R}}/M}$$

This is a d-dimensional real torus with projection map $\pi : M_{\mathbb{R}} \longrightarrow \mathbb{T}$.

We will consider $E \in \mathcal{Q}Con(\mathbb{T})$. For these, $\mu Supp(E) \subset T^*\mathbb{T}$. Note that $T_0\mathbb{T} \cong M_{\mathbb{R}}$ and therefore by homogeneity of $\mathbb{T}$ we have $T^*\mathbb{T} \cong \mathbb{T} \times M_{\mathbb{R}}^{\vee} = \mathbb{T} \times N_{\mathbb{R}}$

# Conormal bundles

We give a standard construction of Lagrangians in $T^*Z$ for a manifold $Z$, and show a related construction which naturally determines half-dimensional subsets of $T^*M_{\mathbb{R}}$ and $T^*\mathbb{T}$ from the fan $\Sigma$.

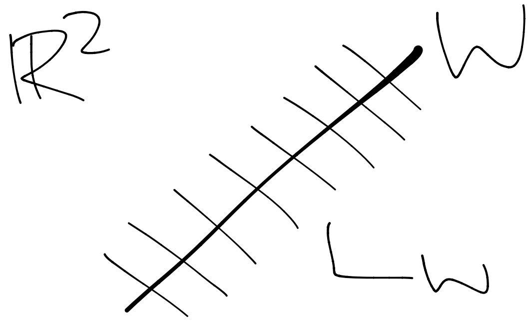
Note: we use $Z$ in place of $M$ for a manifold here, to avoid confusion with the lattice $M$.

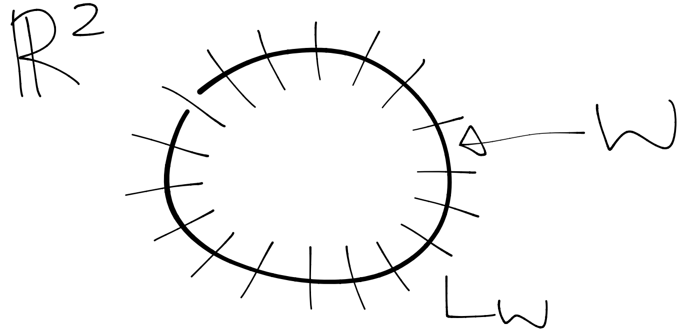
**Def** The conormal bundle $L_W \subset T^*Z$ for a submanifold $W \subset Z$ is

$$L_W = \{(z,\xi) \in T^*Z \mid z \in W,\ \xi(v) = 0 \text{ for all } v \in T_zW\}$$

It can be shown that $L_W$ is Lagrangian.

**Ex** For $Z = \mathbb{R}^2$, we may identify $TZ$ and its dual with the standard inner product. Then conormal bundles may be visualized as follows, in a "whisker diagram". The "whiskers" (this word also describes the long hairs on the face of a cat) show (co)tangent directions.




## Toric skeleta

The following construction is due to Fang-Liu-Treumann-Zaslow, around '10 [F+]

Note $T^*M_{\mathbb{R}} \cong M_{\mathbb{R}} \times M_{\mathbb{R}}^{\vee} = M_{\mathbb{R}} \times N_{\mathbb{R}}$ and the projection map $\pi : M_{\mathbb{R}} \to \mathbb{T}$ induces a projection map $\pi : T^*M_{\mathbb{R}} \to T^*\mathbb{T}$

Recall for a cone $\sigma$ of a fan we defined:

<u>Def</u> (dual cone) $\sigma^\vee = \{m \mid \langle m,n\rangle \geq 0 \ \forall n\in\sigma\} \subset M_{\mathbb{R}}$

Similarly to the conormal bundle, we have

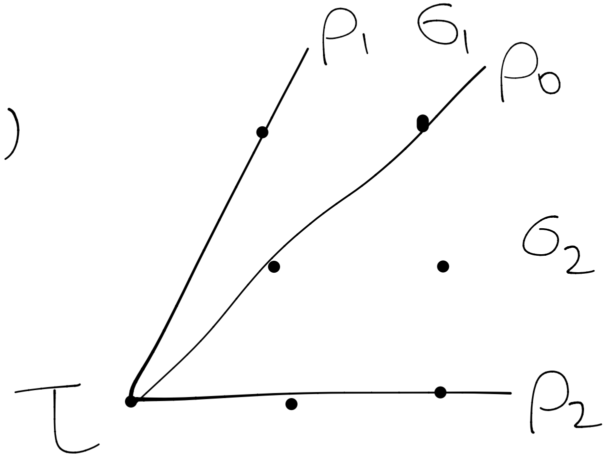
<u>Def</u> $\sigma^\perp = \{m \mid \langle m,n\rangle = 0 \ \forall n\in\sigma\} \subset M_{\mathbb{R}}$

<u>Def</u> For a fan $\Sigma$ we define <u>skeleta</u>

$$\widetilde{\Lambda}_\Sigma = \bigcup_{\sigma\in\Sigma} \sigma^\perp \times \sigma \subset T^* M_{\mathbb{R}}$$

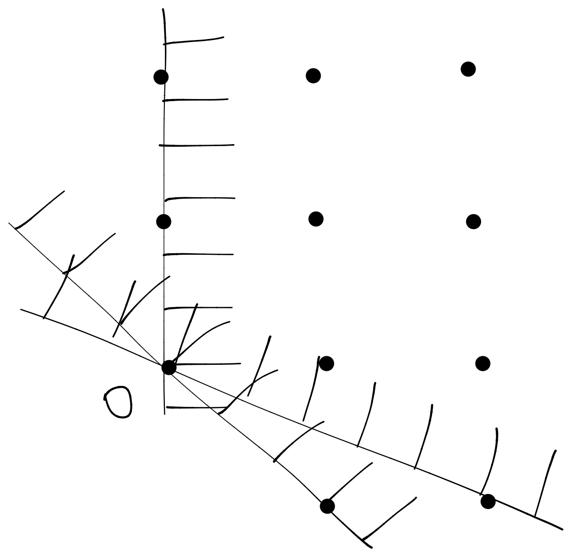
and $\Lambda_\Sigma = \pi(\widetilde{\Lambda}_\Sigma) \subset T^*\mathbb{T}$

<u>Ex</u> Recall that for $\Sigma$ as shown

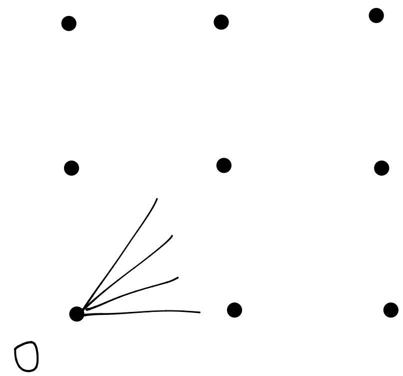



we have $X_\Sigma = \widetilde{\mathbb{C}^2/\mathbb{Z}_2}$, the minimal resolution $\cong$ total space of $\mathcal{O}(-2)$ on $\mathbb{P}^1$

$\widetilde{\Lambda}_\Sigma$ is then the union of the zero section in $T^*M_{\mathbb{R}}$ (from $\tau$) and the following.

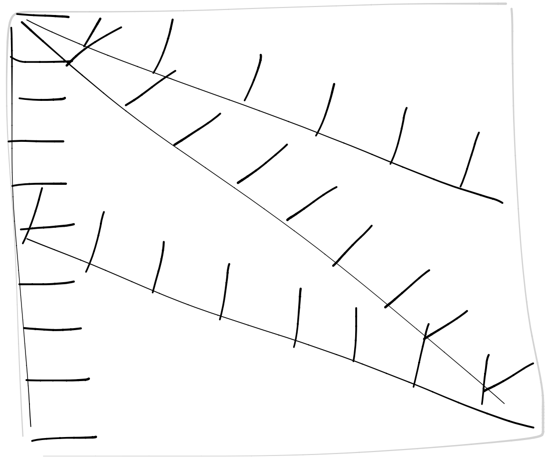



from $\rho$'s

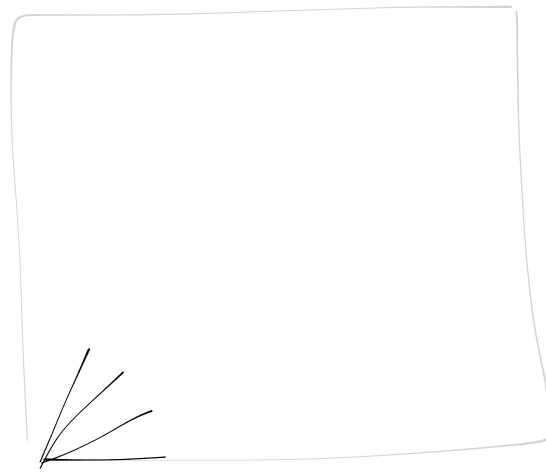



from $\sigma$'s

To sketch $\Lambda_\Sigma \subset T^*\mathbb{T}$ it is convenient to draw $\mathbb{T}$ as follows. A point $l \in [0,1)^2$ below corresponds to an equivalence class $l + M \in \mathbb{T}$. The skeleton $\Lambda_\Sigma$ is then the zero section, union the loci shown.



from $\rho$'s



from $\sigma$'s

**Rem** The canonical bundle $\omega$ for $X_\Sigma$ may be shown to be trivial, by standard toric techniques: hence this example is a (non-compact) Calabi-Yau.

## Toric coherent-constructible correspondence (CCC)

We describe a recent result of Kuwagaki [Ku], which follows work on CCC by many authors, including Bondal and authors FLTZ above.

For toric variety $X_\Sigma$ and skeleton $\Lambda_\Sigma \subset T^*\mathbb{T}$ as above, we have

**Thm** $D(QCoh(X_\Sigma)) \xrightarrow{\sim} D(QCon_{\Lambda_\Sigma}(\mathbb{T}))$

The subscript $\Lambda_\Sigma$ here indicates the full subcategory of sheaves microsupported in that locus.

Let $U_\sigma$ be the open subset of $X_\Sigma$ corresponding to $\sigma \in \Sigma$, and $i$ its embedding. Then the sheaf $i_* \mathcal{O}_{U_\sigma}$ is mapped by the equivalence to $\pi_! \underline{\mathbb{C}}_{Int(\sigma^\vee)}$. Here $\pi_!$ denotes a subsheaf of the pushforward $\pi_*$, namely those sections with "proper support" relative to $\pi: M_\mathbb{R} \to \mathbb{T}$. (As the fibres of $\pi$ are not proper, this makes a difference.)