§4 Functional Analysis w.r.t. pmGH conv.

§§4.1 Case; fixed sp

(X, d, m) : $RCD(K, N)$ sp $(N<\infty)$

Compactness of $H^{1,2}$-fcts

If $f_i \in H^{1,2}(X, d, m)$ $(i=1,2,\cdots)$ satisfy

$$\sup_i \|f_i\|_{H^{1,2}} < \infty,$$

then $\exists f_{i(j)} \xrightarrow{L^2_{loc}\text{-str.}} \exists f \in H^{1,2}$

s.t.

$$\liminf_{j\to\infty} \int_X |Df_{i(j)}|^2 \geq \int_X |Df|^2$$

Pf. This is a direct consequence of BG ineq & $(1,1)$-Poincaré ineq.

Compactness of Laplacian

If $f_i \in D(\Delta)$ $(i=1,2,\cdots,)$ satisfy

$$\begin{cases} \cdot \ \sup_i \|\Delta f_i\|_{L^2} < \infty \\ \cdot \ f_i \to {}^{\exists} f \text{ in } L^2, \end{cases}$$

then $f \in D(\Delta)$, $f_i \to f$ in $H^{1,2}$ &

$\Delta f_i \to \Delta f$ weakly in L^2

Pf.

$$\int |\nabla f_i|^2 = -\int \Delta f_i \cdot f_i \le \underbrace{\|\Delta f_i\|_{L^2}}_{\le C} \cdot \|f_i\|_{L^2}$$

$\Rightarrow f \in H^{1,2}$ & $f_i \to f$ weakly in $H^{1,2}$

• $\Delta f_i \to {}^{\exists}F$ weakly in L^2

$$\int \langle \nabla f, \nabla h\rangle = \lim_{i\to\infty} \int \langle \nabla f_i, \nabla h \rangle$$

$$= \lim_{i\to\infty} -\int \Delta f_i \cdot h = -\int F h$$

$\Rightarrow f \in D(\Delta)$ & $F = \Delta f$

• $$\int |\nabla f_i|^2 = -\int \underbrace{\Delta f_i}_{\text{weak}\ \to\ \Delta f} \cdot \underbrace{f_i}_{\text{strong}\ L^2\ \to\ f} \to -\int \Delta f \cdot f = \int |\nabla f|^2$$

$\Rightarrow f_i \to f$ in $H^{1,2}$ //

Rem.

$\exists$ loc. vers.

§§4.1 Case; pmGH case.

$\exists (Y, d)$, $\exists \varphi_i : X_i \hookrightarrow Y$ s.t.

$\cdot\ (\varphi_i)_\# m_i \to \varphi_\# m$ weak sense

Conv. of fcts.

$\cdot$ $(X_i, d_i, m_i, x_i) \xrightarrow{pmGH} (X, d, m, x)$ $\cdot\ \varphi_i(x_i) \to \varphi(x)$

: RCD(K, N) sps, where $x_i \in X_i$ & $x \in X$

$\cdot$ $f_i : X_i \to \mathbb{R}$ $(i = 1, 2, \cdots)$ & $f : X \to \mathbb{R}$

Then for ——,

$\exists$ L^p-str/weak conv. $(1<p<\infty)$

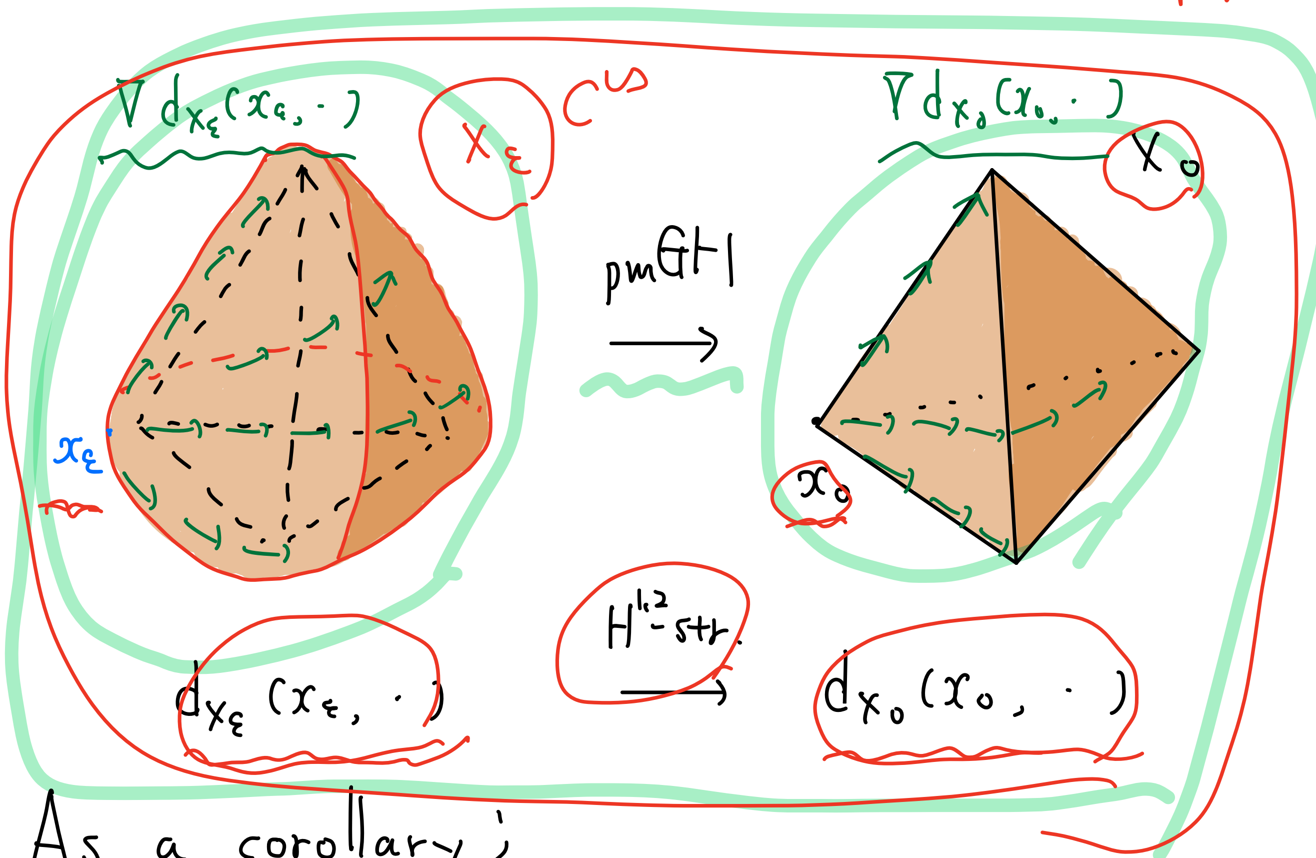
$\sup \|f_i\|_{L^p} < \infty$ & $\int_Y \psi f_i \, d(\varphi_i)_\# m_i \to \int_Y \psi f \, d\varphi_\# m$ for $\forall \psi \in C_{bs}(Y)$

$\exists$ $H^{1,2}$ 〃

$\exists$ Similar results as in §§4.1

Pf. Understand the heat flow via OTT;

Gradient flow of relative entropy



As a corollary;

Spectral conv.

Under (p)mGH of cpt RCD(K,N) sps

eigenvalues/eigenfunctions behave continuously.

Pf.

- $(X_i, d_i, m_i) \to (X, d, m)$ $\quad \int_{X_i} f_i = 0$
- For $\forall f \in H^{1,2}(X)$, $\exists f_i \in H^{1,2}(X_i)$ s.t. $\quad F_i \to f \quad \int_X f = 0 \quad$ in L^2

$f_i \to f$ in $H^{1,2}$ $\quad h_t F_i \to h_t f \quad H^{1,2}_{str}$

$$\Rightarrow \frac{\int_X |Df|^2}{\int_X |f|^2} = \lim_{i\to\infty} \frac{\int |Df_i|^2}{\int_{X_i} |f_i|^2} \geq \limsup_{i\to\infty} \lambda_1(X_i) \quad \text{in } H^{1,2} \quad f$$

$$\Rightarrow \quad \lambda_1(X) \geq \limsup \lambda_1(X_i) \quad 0 = \lambda_0 < \lambda_1$$

- $-\Delta_i f_i = \lambda_1(X_i) f_i \qquad \int_{X_i} |Df_i|^2$

$$\Rightarrow \liminf_{i\to\infty} \lambda_1(X_i) = \liminf_{i\to\infty} \frac{\int_{X_i} |Df_i|^2}{\int_{X_i} |f_i|^2}$$

cptness of $H^{1,2}$

$$\overset{\exists}{\geq} \frac{\int_X |Df|^2}{\int_X |f|^2} \geq \lambda_1(X)$$

$$\frac{\int |Df|^2}{\int_X \left|f - \fint_X f\right|^2}$$

Similarly $\quad \lambda_\ell(X_i) \to \lambda_\ell(X)$

Rem.

$\exists$ loc. vers.

$$\underbrace{\fint_B \left| f - \fint_B f \right|^2 \le r \cdot c \cdot \fint_B |\nabla f|^2}_{(f \not\approx \varphi)}$$

Harmonicity under blow-up

- (X, d, m) : $RCD(K, N)$ sp.
- $f : B_s(x) \longrightarrow \mathbb{R}$ with $\Delta f \in L^\infty$

$\Rightarrow f$: locally Lip

Then $\forall r_i \downarrow 0$

RCD(0,N)

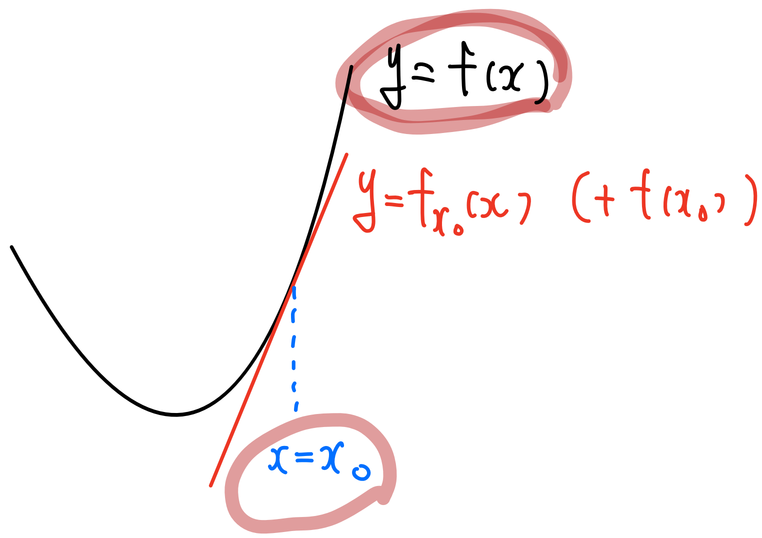
$$\exists \left(X, \frac{d}{r_{i(j)}}, \frac{m}{m(B_{r_{i(j)}}(x))}, x\right) \xrightarrow{\text{pmGH}} \exists\, (Y, d_Y, m_Y, y)$$

tangent cone at x

$\exists f_x : Y \longrightarrow \mathbb{R}$: harm. & Lip

s.t. $$\frac{f - f(x)}{r_{i(j)}} \xrightarrow{H^{1,2}_{loc}\text{-str.}} f_x$$

Pf. $$\left|\Delta_{i(j)}\left(\frac{f - f(x)}{r_{i(j)}}\right)\right| = |r_i \Delta f| \le r_i \cdot \exists C \downarrow 0$$



Approximation by harm. fcts.

- $(X_i, d_i, m_i, x_i) \xrightarrow{pmGH} (X, d, m, x)$

  : $RCD(K,N)$

- $f : B_r(x) \to \mathbb{R}$: harm

Then $\forall\, \underline{s < r}$ sharp

$\exists f_{i(j)} : B_s(x_{i(j)}) \to \mathbb{R}$: harm.

s.t. $f_{i(j)} \to f$: $H^{1,2}$-str on $B_s(x)$

Pf.

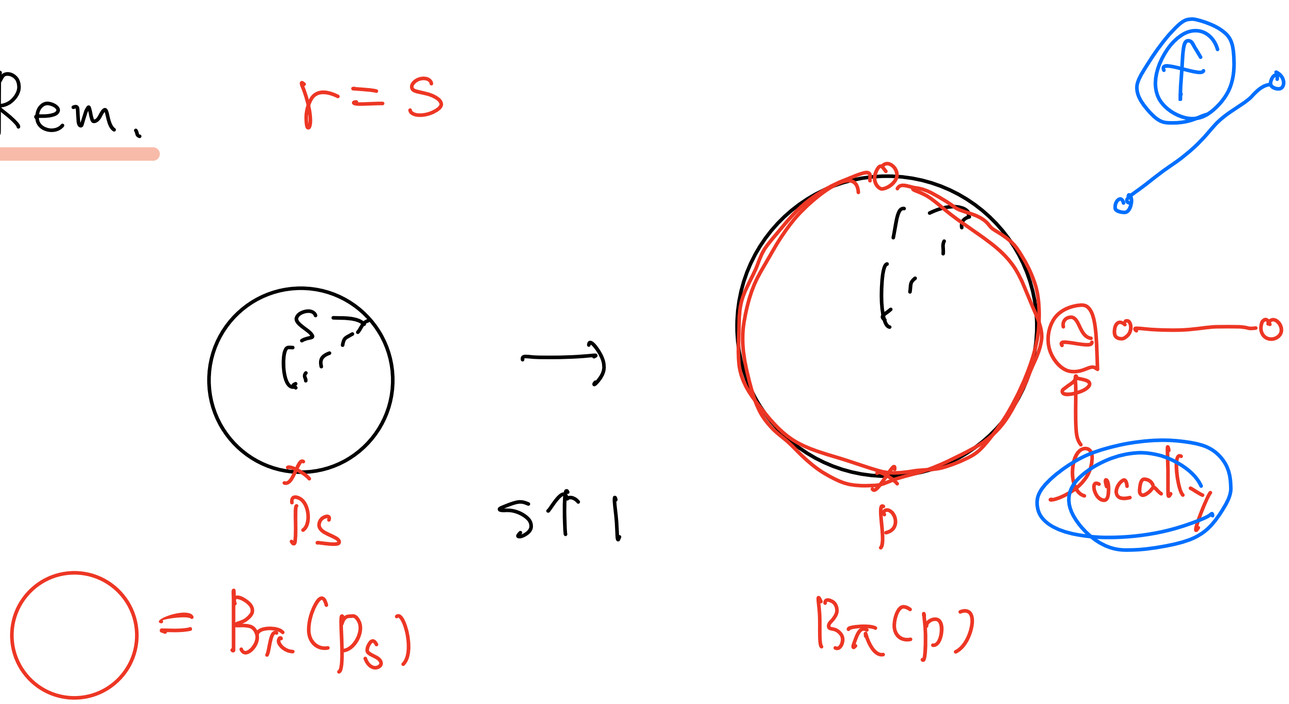
- $\exists\, \bar{f}_i \longrightarrow f$: $H^{1,2}$-str on $B_s(x)$
- $\exists\, f_i : B_s(x_i) \longrightarrow \mathbb{R}$: harm. s.t.

  $f_i = \bar{f}_i$ on $\partial B_s(x_i)$ $\quad$ $f_i - \bar{f}_i \in H^{1,1}_0(B_s(x_i))$
- Applying local spectral

  this holds for $\mathcal{L}^1$-a.e $s \in (0,\infty)$

$\Rightarrow$ Conclusion!

Rem. $\quad r = s$



§5 Density of regular set

(X, d, m) : $RCD(K, N)$ sp. $(N < \infty)$

$X \neq pt$

Regular set

x : ℓ-dim. regular $(x \in \mathcal{R}_\ell)$

$\overset{\text{def}}{\iff}$ $\forall$ tangent cone at x is ℓ-dim. Eucl.

$$\mathcal{R} := \bigcup_\ell \mathcal{R}_\ell$$

Goal

$$m(X \setminus \mathcal{R}) = 0$$

In particular $\mathcal{R}$ is dense.

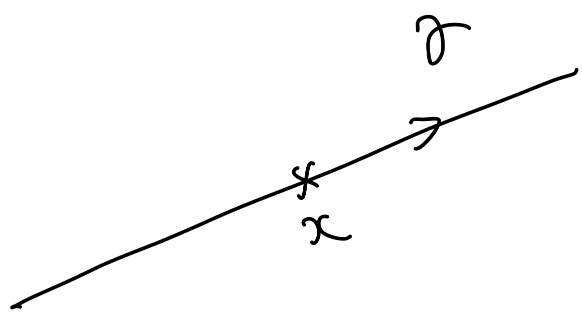
Rem.

$\exists$ example s.t $X \setminus R$ is also dense.

Step 1 of pf. of Goal

For m-a.e. $x \in X$, x is a mid point of a geodesic.

pf. Use OTT //



Step 2 of pf. of Goal

- $\mathcal{E}_\ell = \mathcal{E}_\ell(X) := \{x \in X \mid {}^\forall T_x X \cong \mathbb{R}^\ell \times {}^\exists Z\}$

  harm. $\to \mathbb{R}^\ell$

- $\dim B := \max\{\ell \in \mathbb{N} \mid m(B \cap \mathcal{E}_\ell) > 0\}$

  for ${}^\forall B \subset X$ with $m(B) > 0$.

Then
$$\liminf_{i \to \infty} \dim B_i \geq \dim B$$
whenever
$$I_{B_i} \xrightarrow{\text{loc-str.}} I_B$$

Pf.

Use results in §9.

Step 3 of pf. of Goal

$$\dim B \geq \dim T_y X$$

for $^\forall B \subset X$ with $m(B) > 0$

$^\forall y \in \mathrm{Leb}\, B = \left\{ x \in B \;\middle|\; \frac{m(B_r(x) \cap B)}{m(B_r(x))} \to 1 \text{ as } r \to 0 \right\}$

$(m(B \setminus \mathrm{Leb}\, B) = 0)$

$^\forall T_y X$: tang. cone at y.

Pf. Applying step2 to

$$1_B \to 1_{T_y Y} = 1 \;: L^2_{loc}\text{-str}$$

w.r.t.

$$\left(X, \frac{d}{r_i}, \frac{m}{m(B_{r_i}(y))}, y\right) \to (T_y Y, \cdots)$$

//

Step 4 of pf. of Goal

$R_e \subset \mathcal{E}_e$

$$B \cap \mathcal{E}_\ell \sim B \cap R_\ell$$

for $\left\{ \begin{array}{l} \cdot\ {}^\forall B \subset X \text{ with } m(B) > 0 \\ \cdot\ \ell = \dim B \\ \cdot\ A \sim B \overset{\text{dep}}{\iff} m(A \setminus B) + m(B \setminus A) = 0 \end{array} \right.$

Pf.

- If not, then

  $m(B \cap (\mathcal{E}_\ell \setminus R_\ell)) > 0$

- Step3 $\Rightarrow$ $\ell = \dim$ ____ $\geq \dim T_y X$

  for ${}^\forall y \in \text{Leb}$ ____ & ${}^\forall T_y X$

- $T_y X \cong \mathbb{R}^\ell \times Z$ s.t. ${}^\exists Z \neq pt$

  because of $y \in$ ____

- Step1 $\Rightarrow \dim Z \geq 1$

  $\Rightarrow \dim T_y X \geq \ell + 1$

  $\Rightarrow$ Contradiction!

  because ____

Final step of Goal

$$m(X \setminus R) = 0$$

Actually $m(X \setminus R_{k_1}) = 0$

Pf.

- $k_1 = \dim(X)$
- Assume $m(X \setminus R_{k_1}) > 0$
- $k_2 = \dim(X \setminus R_{k_1})$ $\quad (k_2 \leq k_1)$
- $k_2 < k_1$ because $(X \setminus R_{k_1}) \cap \mathcal{E}_{k_2} \sim (X \setminus R_{k_1}) \cap R_{k_2}$ have positive m-meas.
- Assume $m(X \setminus R_{k_1} \setminus R_{k_2}) > 0$
- $k_3 = \dim(X \setminus R_{k_1} \setminus R_{k_2})$
- .... $\Rightarrow$ Conclusion!