Square-tiled surfaces and interval exchanges: geometry, dynamics, combinatorics and applications

Lecture 12. Sum of Lyapunov exponents of the Kontsevich–Zorich cocycle: outline of the proof.

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Main Theorem and outline of the proof

Main Theorem

Theorem (A. Eskin, M. Kontsevich, A. Zorich, 2014). *The Lyapunov exponents λ_i of the bundle $H^1_{\mathbb{R}}$ along the Teichmüller flow restricted to an $\mathrm{SL}(2,\mathbb{R})$-invariant suborbifold $\mathcal{L} \subseteq \mathcal{H}_1(d_1,\dots,d_n)$ satisfy:*

$$\lambda_1 + \lambda_2 + \dots + \lambda_g \;=\; \frac{1}{12}\cdot\sum_{i=1}^{n}\frac{d_i(d_i+2)}{d_i+1} + \frac{\pi^2}{3}\cdot c_{area}(\mathcal{L})\,.$$

where $c_{area}(\mathcal{L})$ is the Siegel–Veech constant. The top exponent λ_1 is equal to one, $\lambda_1 = 1$.

Hodge norm

The space of holomorphic 1-forms is endowed with a natural Hermitian metric:

$$\langle \omega_1, \omega_2 \rangle := \frac{i}{2} \int_C \omega_1 \wedge \overline{\omega_2}, \qquad \omega_1, \omega_2 \in H^{1,0} = \Gamma(C, \Omega^1_{hol})$$

This metric induces Euclidean metric in $H^1(C, \mathbb{R})$. Namely, for any cohomology classes $c_1, c_2 \in H^1(C, \mathbb{R})$ one can find $\omega_1, \omega_2 \in H^{1,0}$ such that $c_1 = \mathrm{Re}\, \omega_1$, $c_2 = \mathrm{Re}\, \omega_2$. We define $\langle c_1, c_2 \rangle := \mathrm{Re} \langle \omega_1, \omega_2 \rangle$.
The corresponding norm $\|c_1\|^2 := \langle c_1, c_1 \rangle = \frac{i}{2} \int_C \omega_1 \wedge \overline{\omega_1}$ is called the *Hodge norm*; it *is not* preserved by Gauss—Manin connection.

M. Kontsevich found a very convenient expression for the Hodge norm of a vector $L := v_1 \wedge \dots \wedge v_g$ in the exterior power $\Lambda^g H^1(C, \mathbb{R})$ when L represents a Lagrangian subspace:

$$\|L\|^2 := \frac{|v_1 \wedge \dots \wedge v_g \wedge \omega_1 \wedge \dots \wedge \omega_g| \cdot |v_1 \wedge \dots \wedge v_g \wedge \bar{\omega}_1 \wedge \dots \wedge \bar{\omega}_g|}{|\omega_1 \wedge \dots \wedge \omega_g \wedge \bar{\omega}_1 \wedge \dots \wedge \bar{\omega}_g|}\,.$$

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Kontsevich formula

Lemma (M. Kontsevich, 1997). *For any* $L \in \Lambda^g H^1(C, \mathbb{R})$ *representing a Lagrangian subspace and for any basis* $\{\omega_k\}$ *of local holomorphic sections of the Hodge bundle* $H^{1,0}$*, one has:*

$$\Delta_{Teich} \log \|L\| = -\frac{1}{2} \Delta_{Teich} \log \det \langle \omega_i, \omega_j \rangle \, ,$$

where Δ_{Teich} *is the hyperbolic Laplacian along the Teichmüller disc.*

Note that the above expression does not depend neither on the choice of the decomposable vector L nor on the choice of the basis!

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Theorem (M. Kontsevich – idea, 1996; G. Forni – proof, 2002). *Let* $\mathcal{M}$ *be any closed connected* $\mathrm{SL}(2,\mathbb{R})$*-invariant suborbifold of some stratum of Abelian differentials in genus* g*. Let* ν *be the associated linear ergodic probability measure. The top* g *Lyapunov exponents of the Hodge bundle* $H^1_{\mathbb{R}}$ *over* $\mathcal{M}$ *along the Teichmüller flow satisfy the following relation:*

$$\lambda_1 + \dots + \lambda_g = \int_{\mathcal{M}} \Lambda(S)\, d\nu(S)\,.$$

Determinant of Laplace operator

To define a determinant of Laplace operator $\det \Delta_g$ of a smooth Riemannian metric g on a closed nonsingular Riemann surface C one defines the following *spectral zeta function*:

$$\zeta(s) = \sum_{\theta} \theta^{-s} \, ,$$

where the sum is taken over nonzero eigenvalues of Δ_g. This sum converges for $\mathrm{Re}(s) > 0$. The function $\zeta(s)$ should be analytically continued to $s = 0$ and then one defines

$$\log \det \Delta_g := -\zeta'(0) \, .$$

Note that interpreting Δ_g as a $n \times n$-matrix and interpreting θ's as n eigenvalues of this matrix, the definition above gives the usual logarithm of determinant of the matrix.

Analytic Riemann–Roch Theorem

Smoothen conical singularities of a flat metric and define $\det \Delta_{flat}(S, S_0)$ as a limit of the ratio of determinants of Laplacians for such smoothed metrics on flat surfaces S and S_0 in the same stratum.

Theorem (A. Eskin, M. Kontsevich, A. Zorich, 2014). *For any S in any stratum $\mathcal{H}_1(m_1, \dots, m_n)$ of Abelian differentials the following formula holds:*

$$\Delta_{Teich} \log |\det\langle\omega_i, \omega_j\rangle| =$$

$$= \Delta_{Teich} \log \det \Delta_{flat}(S, S_0) - \frac{1}{3}\sum_{j=1}^{n} \frac{m_j(m_j+2)}{m_j+1} .$$

This Theorem was developed by numerous authors in different context. To give a very partial credit we would like to cite A. Belavin and V. Knizhnik; J.-M. Bismut and J.-B. Bost; J.-M. Bismut, H. Gillet and C. Soulé; D. Quillen; L. Takhtadzhyan and P. Zograf. The statement above is initially based on the results of J. Fay. An alternative later and much simpler proof is based on the paper of A. Kokotov and D. Korotkin.

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Polyakov formula

Let g_1 and g_2 be two nonsingular metrics of unit area in the same conformal class on a closed surface C. Let K_{g_1} be the curvature of the metric g_1. Let a smooth function 2ϕ be the logarithm of the conformal factor relating the two metrics: $g_2 = \exp(2\phi) \cdot g_1$.

Polyakov Formula

$$\begin{aligned}\log\det\Delta_{g_2} - \log\det\Delta_{g_1} \; &= \\ &= \frac{1}{12\pi}\left(\int_C \phi\,\Delta_{g_1}\phi\, dg_1 \;-\; 2\int_C \phi\, K_{g_1}\, dg_1\right) .\end{aligned}$$

Compactification Theorem

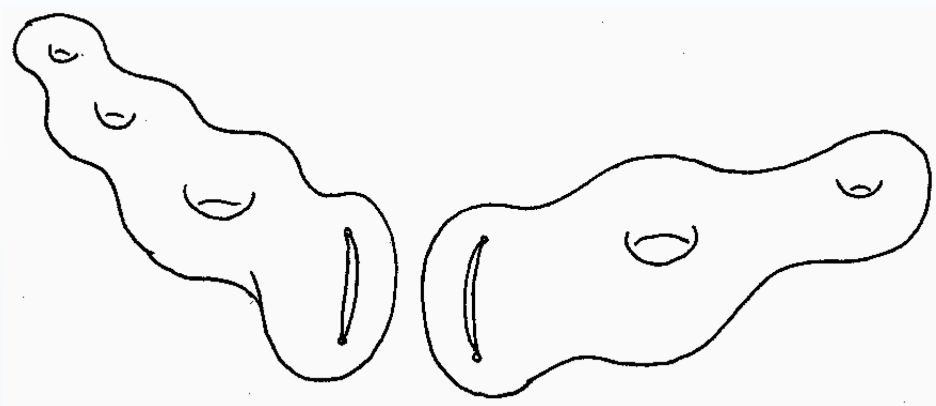
Theorem (A. Eskin, M. Kontsevich, A. Zorich, 2014). *Consider a sequence of flat surfaces $S_j = (C_j, \omega_j)$, where holomorphic 1-forms ω_j stay in a fixed stratum $\mathcal{H}(m_1, \dots, m_n)$. Suppose that the underlying Riemann surfaces C_j converge to a stable curve C_∞ and let $\tilde{C}_\infty$ be an irreducible component of C_∞. Let Y_j be the corresponding component in a thick-thin decomposition of (C_j, ω_j).*

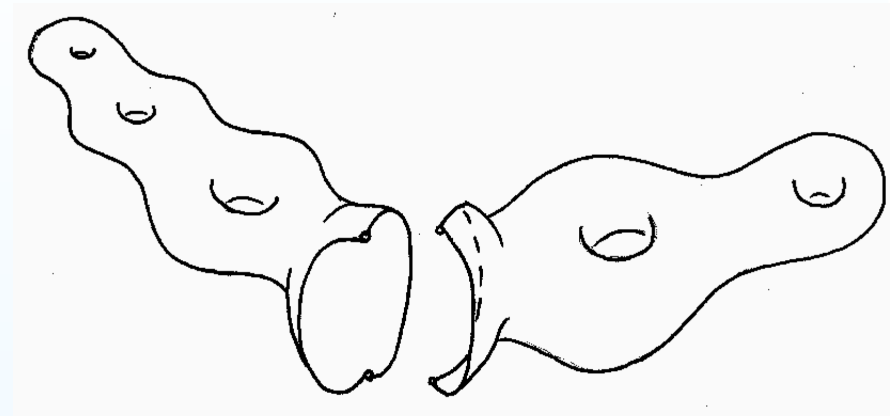
Normalizing $\omega_j|_{Y_j}$ by an appropriate scalar $\ell(Y_j, \omega_j)$, and passing to an appropriate subsequence, we get a sequence which tends to a nonzero meromorphic 1-form $\tilde{\omega}$ on the irreducible component $\tilde{C}_\infty$.

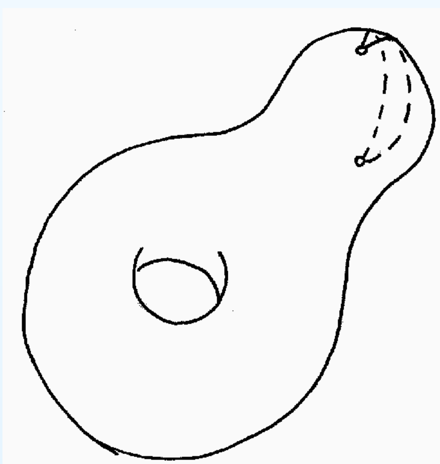
It follows from the results of K. Rafi, that the scalar $\ell(Y_j, \omega_j)$ has an explicit geometric meaning: it is the length of the shortest (in the flat metric) essential closed curve on the thick component Y_j.

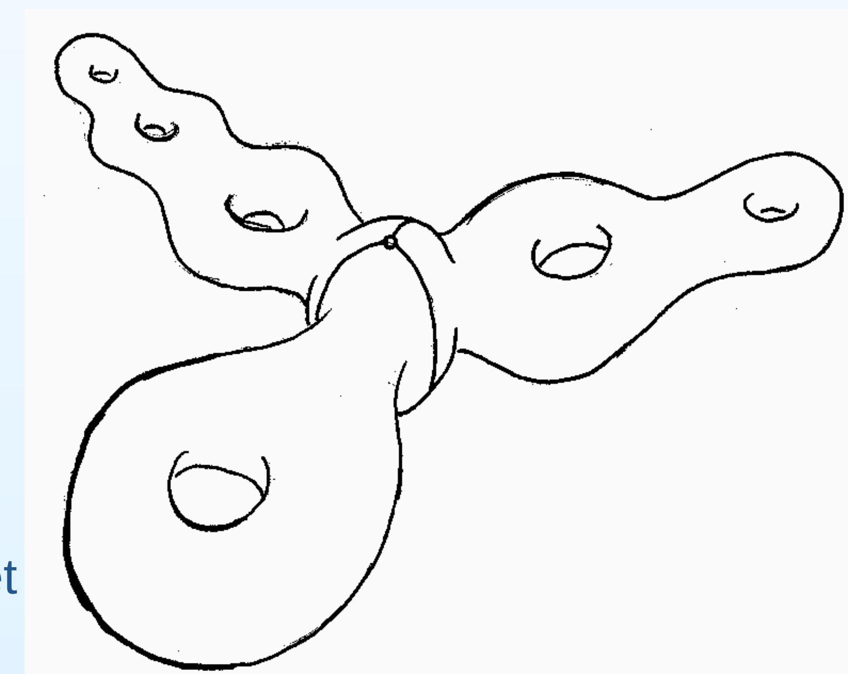
Remark. The above Theorem was recently enhanced by M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky and M. Möller to a multiscale compactification with a detailed combiatorial description of all possible hierarchies of degenerations.

Nonexisting stable curve







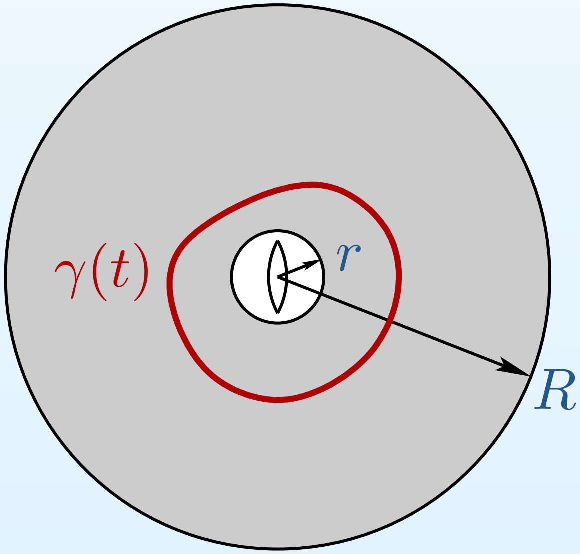


It might seem that contracting the cuts we get a stable curve... with a *triple* intersection!

Invisible $\mathbb{CP}^1$

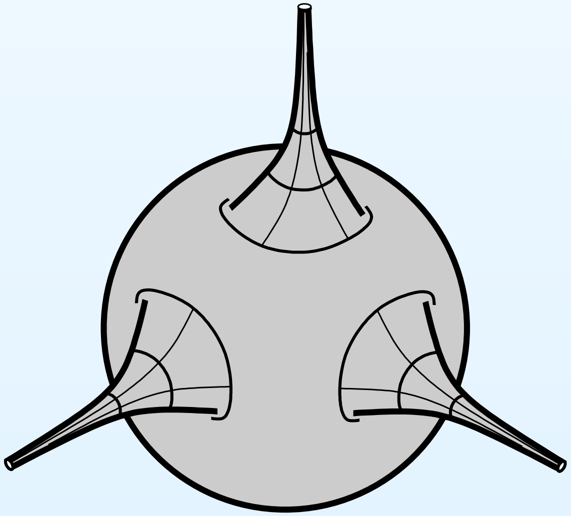
The flat metric does not see one more component $\tilde{C}_\infty$ of the underlying stable curve C_∞.

Every tiny slit of vanishing length t can be encircled by a flat annulus of modulus $M(t) = \frac{1}{2\pi}\log\frac{R}{r}$, so $M(t) \to +\infty$ as $t \to 0$. The unique hyperbolic metric in the conformal class of the original flat metric thus inherits an annulus of modulus $M(t) \to +\infty$. By a result of S. Wolpert this hyperbolic annulus contains a hyperbolic geodesic $\gamma(t)$ of length $\ell(t)$ satisfying $\ell(t) \cdot M(t) \to \pi$, so this hyperbolic geodesic $\gamma(t)$ gets pinched when the slit is contracted.



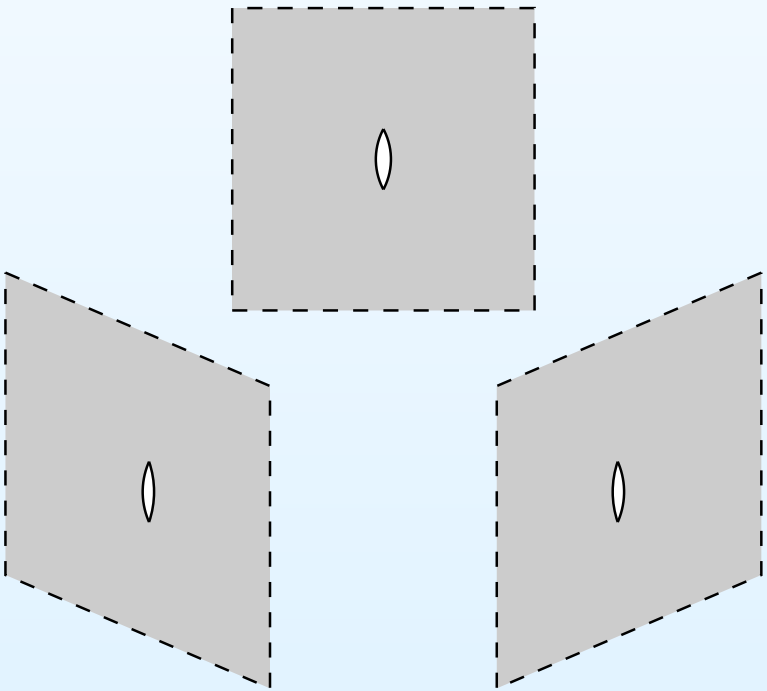
Invisible $\mathbb{CP}^1$

We have observed that each of the three annuli around the slits contains a pinching hyperbolic geodesic. Cutting the hyperbolic surface by these three closed geodesics we chop off the three original thick components and get one more thin component in the middle. This forth component is $\mathbb{CP}^1$ with two mark points and with three “almost cusps” bordered by the short closed hyperbolic geodesics by which we chopped off the thick components. In the limit they degenerate to three nodal points by which the three thick components are attached to the thin component $\tilde{C}_\infty = \mathbb{CP}^1$ (of vanishing flat area) in the middle.



Invisible $\mathbb{C}\mathrm{P}^1$

Cutting out the corresponding thin component and rescaling the flat metric on it to keep the length of the saddle connection equal to one, we get a well-defined meromorphic 1-form on the limiting $\mathbb{C}\mathrm{P}^1$. The corresponding flat $\mathbb{C}\mathrm{P}^1$ can be constructed by cyclically gluing three copies of a flat plane with a unit slit. The endpoints of the slits give rise to two conical points each of cone angle 6π (the two zeroes of degree 2). The point at the absolute of each of the three open planes is a pole of order 2. Thus, $\tilde{\omega} \in \mathcal{H}(2, 2, -2, -2, -2)$.



Comparing determinants of flat and hyperbolic Laplacians

Combining the Polyakov formula with the Compactification Theorem we get

Theorem (A. Eskin, M. Kontsevich, A. Zorich, 2014). *Let $\ell(S)$ be the length of the shortest saddle connection on a flat surface S. Then*

$$|\log\det\Delta_{\mathit{flat}}(S,S_0) - \log\det\Delta_{Teich}(S,S_0)| = O(|\log\ell(S)|)$$

Idea of the proof: First prove it when a sequence S_τ of flat surfaces tends to a limiting object: underlying Riemann surfaces tend to a stable curve, and restriction of the flat metric to each thick component tends (after rescaling) to a limiting meromorphic 1-form.

The rest follows by contradiction: it is sufficient to use the Compactification Theorem by passing to subsequences.

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Cutoff near the boundary of a stratum

Using results of J. Jorgenson and R. Lundelius on asymptotics of $\det\Delta_{Teich}(S,S_0)$ when a Riemann surface degenerates, we get

Corollary (A. Eskin, M. Kontsevich, A. Zorich, 2014). *For a flat surface close to a boundary of the stratum one has*

$$\log\det\Delta_{\mathit{flat}}(S,S_0) = -\frac{\pi}{3}\sum\frac{h_i}{w_i} + O(|\log\ell(S)|)\,,$$

where summation is performed over all long and narrow cylinders. Here w_i denotes the (short) perimeter and h_i — a (long) height of a cylinder.

Using the Green formula to replace $\int\Delta_{Teich}\log\det\Delta_{\mathit{flat}}\,d\nu$ by an appropriate integral over a hypersurface close to the boundary combined with our asymptotics for $\log\det\Delta_{\mathit{flat}}$ one gets exactly the same integral as the one representing the Siegel—Veech constant. The Main Theorem is proved. $\square$

Zoom on the proof of the initial formula for the sum of Lyapunov exponents

Hodge norm of a "Lagrangian subspace"

Consider the map $\Omega : \Lambda^{2g} H^1(S, \mathbb{C}) \to \mathbb{C}/\pm$ given by $\Omega(\lambda) := \lambda(e_1, \ldots, e_{2g})$, where $\{e_1, \ldots, e_{2g}\}$ is any $\mathbb{Z}$-basis for $H_1(S, \mathbb{Z})$.

Let $L = v_1 \wedge \cdots \wedge v_g$ be a decomposable vector in $\Lambda^g H^1(S, \mathbb{R})$ representing a Lagrandian subspace. Let $\omega_1, \ldots, \omega_g$ form a basis in $H^{1,0}(S)$. We define

$$\|L\|^2 := \frac{|\Omega(v_1 \wedge \cdots \wedge v_g \wedge \omega_1 \wedge \cdots \wedge \omega_g)| \cdot |\Omega(v_1 \wedge \cdots \wedge v_g \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_g)|}{|\Omega(\omega_1 \wedge \cdots \wedge \omega_g \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_g)|}$$

Clearly, this definition does not depend on a choice of the basis in $H^{1,0}(S)$.

Note that $\Omega(\omega_1 \wedge \cdots \wedge \omega_g \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_g) = \det\langle \omega_i, \omega_j \rangle$, where

$$\langle \omega_i, \omega_j \rangle = \begin{pmatrix} \langle \omega_1, \omega_1 \rangle & \ldots & \langle \omega_1, \omega_g \rangle \\ \ldots & \ldots & \ldots \\ \langle \omega_g, \omega_1 \rangle & \ldots & \langle \omega_g, \omega_g \rangle \end{pmatrix}$$

is the matrix of Hermitian scalar product of elements of the basis in $H^{1,0}(S)$.

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Kontsevich formula

Proposition (Kontsevich, 1997). *For any flat surface S, any decomposable vector L in the exterior power $\Lambda^g H^1(S,\mathbb{R})$ representing a Lagrangian subspace and for any basis $\{\omega_k\}$ of local holomorphic sections of the Hodge vector bundle $H^{1,0}$ one has:*

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Proof. $2\Delta_{Teich} \log \|L\| = \Delta_{Teich} \log \|L\|^2 =$

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$$= \Delta_{Teich} \log |\Omega(v_1\wedge\dots\wedge v_g\wedge\omega_1\wedge\dots\wedge\omega_g)| +$$

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Note that $v_1,\dots,v_g$ do not change along the Teichmüller disc, so the first function is a holomorphic function of the deformation parameter, and the second one is an antiholomorphic. Hence both functions are harmonic. $\square$

Background Theorem

Denote

$$\Lambda(S) := -\frac{1}{4}\Delta_{Teich} \log |\det\langle\omega_i,\omega_j\rangle| \,,$$

where Δ_{Teich} is the hyperbolic Laplacian along the Teichmüller disc in the metric of constant negative curvature -4. We consider $\Lambda(S)$ as a scalar function on $\mathrm{P}\mathcal{H}(m_1,\dots,m_n)$, or, when convenient, we pull it back to $\mathcal{H}_1(m_1,\dots,m_n)$.

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Uniform bounds and Green formula

Lemma (G. Forni, 2002). *For any flat surface S in any stratum in any genus one has:*

$$\max_{\substack{c\in H^1(S,\mathbb{R})\text{such}\\ \text{that } \|c\|=1}}\left|\frac{d\|c\|}{dt}\right| \le 1\,, \qquad \max_{\substack{L\in\Lambda^g(H^1(S,\mathbb{R}))\\ L\neq 0}}\left|\frac{d\log\|L\|}{dt}\right| \le g\,, \qquad |\Lambda(S)| \le g\,.$$

Proof. These uniform estimates are direct corollaries of the Rauch-type formulas proved by G. Forni. □

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Uniform bounds and Green formula

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Proof of the Background Theorem

Consider the bundle $\mathcal{G}r_g(\mathcal{M})$ of Grassmannians $\mathcal{G}r_g(\mathbb{R}^{2g})$ associated to the Hodge vector bundle $H^1_{\mathbb{R}}$ over $\mathcal{M}$. Its fiber over a “point” $S \in \mathcal{M}$ can be naturally identified with the set of decomposable vectors of norm one in $\Lambda^g H^1(S,\mathbb{R})$.

The sum of the top k Lyapunov exponents of a vector bundle is equal to the top Lyapunov exponent of its k-th exterior power.

Denote by $d\sigma_S$ the normalized Haar measure in the fiber of the Grassmannian bundle over a point $S \in \mathcal{M}$. For almost all pairs (S,L) where $S \in \mathcal{M}$, and $L \in \mathcal{G}r_g\left(H^1(S,\mathbb{R})\right)$ one has

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We start with the identity

$$\lambda_1+\cdots+\lambda_g = \lim_{T\to+\infty} \frac{1}{T} \log \|L(g_t S)\| = \lim_{T\to+\infty} \frac{1}{T} \int_0^T \frac{d}{dt} \log \|L(g_t S)\| \, dt$$

Averaging it over the Grassmanian and over the initial “point” S we get

$$\lambda_1 + \cdots + \lambda_g = \int_{\mathcal{G}r_g(\mathcal{M})} \lim_{T\to+\infty} \frac{1}{T} \int_0^T \frac{d}{dt} \log \|L(g_t S)\| \, dt \, d\nu \, d\sigma_S$$

Apply an extra averaging over the circle

$$\lambda_1 + \cdots + \lambda_g = \int_{\mathcal{G}r_g(\mathcal{M})} \lim_{T \to +\infty} \frac{1}{T} \int_0^T \frac{d}{dt} \log \|L(g_t S)\| \, dt \, d\nu \, d\sigma_S$$

$$= \int_0^{2\pi} \int_{\mathcal{G}r_g(\mathcal{M})} \lim_{T \to +\infty} \frac{1}{T} \int_0^T \frac{1}{2\pi} \frac{d}{dt} \log \|L(g_t r_\theta S)\| d\theta \; dt \, d\nu \, d\sigma_S$$

Using the uniform bounds interchange the limit with the integral over the circle.

$$\lambda_1 + \cdots + \lambda_g = \int_{\mathcal{G}r_g(\mathcal{M})} \lim_{T \to +\infty} \frac{1}{T} \int_0^T \frac{d}{dt} \log \|L(g_t S)\| \, dt \, d\nu \, d\sigma_S$$

$$= \int_{\mathcal{G}r_g(\mathcal{M})} \lim_{T \to +\infty} \frac{1}{T} \int_0^T \frac{1}{2\pi} \int_0^{2\pi} \frac{d}{dt} \log \|L(g_t r_\theta S)\| d\theta \; dt \, d\nu \, d\sigma_S$$

Apply the Green formula in the second line to pass to the third line.

$$\lambda_1 + \cdots + \lambda_g = \int_{\mathcal{G}r_g(\mathcal{M})} \lim_{T\to+\infty} \frac{1}{T} \int_0^T \frac{d}{dt} \log \|L(g_t S)\| \, dt \, d\nu \, d\sigma_S$$

$$= \int_{\mathcal{G}r_g(\mathcal{M})} \lim_{T\to+\infty} \frac{1}{T} \int_0^T \frac{1}{2\pi} \int_0^{2\pi} \frac{d}{dt} \log \|L(g_t r_\theta S)\| d\theta \;\, dt \, d\nu \, d\sigma_S$$

$$= \int_{\mathcal{G}r_g(\mathcal{M})} \lim_{T\to+\infty} \frac{1}{T} \int_0^T \frac{\tanh(t)}{2|D_t|} \int_{D_t} \Delta_{Teich} \log \|L(g_t r_\theta S)\| dg_{hyp} \;\, dt \, d\nu \, d\sigma_S$$

Applying the Kontsevich formula we get an expression independent of L:

$$\lambda_1 + \cdots + \lambda_g = \int_{\mathcal{G}r_g(\mathcal{M})} \lim_{T\to+\infty} \frac{1}{T} \int_0^T \frac{d}{dt} \log \|L(g_t S)\| \, dt \, d\nu \, d\sigma_S$$

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$$= \int_{\mathcal{M}} \lim_{T\to+\infty} \frac{1}{T} \int_0^T \frac{\tanh(t)}{2|D_t|} \int_{D_t} -\frac{1}{2} \Delta_{Teich} \log |\det\langle \omega_i, \omega_j \rangle| dg_{hyp} \; dt \, d\nu$$

Passage to the line five is nothing but application of the notation $\Lambda(S)$.

$$\lambda_1 + \cdots + \lambda_g = \int_{\mathcal{G}r_g(\mathcal{M})} \lim_{T\to+\infty} \frac{1}{T} \int_0^T \frac{d}{dt} \log \|L(g_t S)\| \, dt \, d\nu \, d\sigma_S$$

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$$= \int_{\mathcal{M}} \lim_{T\to+\infty} \frac{1}{T} \int_0^T \frac{\tanh(t)}{|D_t|} \int_{D_t} \Lambda(S) \, dg_{hyp} \; dt \, d\nu$$

Change the order of integration using the uniform bound.

$$\lambda_1 + \cdots + \lambda_g = \int_{\mathcal{G}r_g(\mathcal{M})} \lim_{T\to+\infty} \frac{1}{T} \int_0^T \frac{d}{dt} \log \|L(g_t S)\| \, dt \, d\nu \, d\sigma_S$$

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$$= \int_{\mathcal{M}} \Lambda(S) \, d\nu \cdot \left(\lim_{T\to+\infty} \frac{1}{T} \int_0^T \tanh(t) \, dt \right)$$

The very last equality is an elementary property of $\tanh(t)$.

$$\lambda_1 + \cdots + \lambda_g = \int_{\mathcal{G}r_g(\mathcal{M})} \lim_{T\to+\infty} \frac{1}{T}\int_0^T \frac{d}{dt}\log\|L(g_t S)\|\,dt\,d\nu\,d\sigma_S$$

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$$= \int_{\mathcal{M}} \Lambda(S)\,d\nu\cdot\left(\lim_{T\to+\infty}\frac{1}{T}\int_0^T \tanh(t)\,dt\right) = \int_{\mathcal{M}} \Lambda(S)\,d\nu(S)\,. \qquad \square$$