Motivic Galois theory for algebraic Mellin transforms

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Online seminar on periods and motives
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Plan

1. Algebraic Mellin transforms
2. Twisted cohomology
3. Application to Feynman integrals

Algebraic Mellin transforms

(Not in this talk) The classical Mellin transform (Mellin, 1897)

$$\varphi : (0,\infty) \to \mathbb{C} \qquad \rightsquigarrow \qquad (\mathcal{M}\varphi)(s) = \int_0^\infty x^s \varphi(x) \frac{dx}{x}.$$

Algebraic Mellin transforms (Aomoto, 1974)

$$I(s) = \int_\sigma f^s \omega.$$

- X an (affine, smooth) algebraic variety over a field $k \subset \mathbb{C}$.
- $f : X \to \mathbb{G}_m$ an invertible function on X.
- ω an algebraic differential form on X, σ a topological cycle on X.

(Bloch–Vlasenko call them “motivic Mellin transforms” or “motivic Γ-functions”.)

More generally, for $f = (f_1, \ldots, f_N) : X \to \mathbb{G}_m^N$, consider multivariate versions:

$$I(s_1, \ldots, s_N) = \int_\sigma f_1^{s_1} \cdots f_N^{s_N} \omega.$$

- The beta function:

$$\mathrm{B}(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} = \int_0^1 x^s(1-x)^t \frac{dx}{x(1-x)} .$$

Corresponds to $(x, 1-x) : \mathbb{P}^1 \setminus \{\infty, 0, 1\} \longrightarrow \mathbb{G}_m^2$.

Examples of algebraic Mellin transforms

- The beta function:

$$\mathrm{B}(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} = \int_0^1 x^s(1-x)^t \frac{dx}{x(1-x)} \, .$$

Corresponds to $(x, 1-x) : \mathbb{P}^1 \setminus \{\infty, 0, 1\} \longrightarrow \mathbb{G}_m^2$.

- The classical hypergeometric function:

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \quad \text{where } (t)_n = t(t+1)\cdots(t+n-1).$$

$$\mathrm{B}(b, c-b)\,{}_2F_1(a,b,c;z) = \int_0^1 x^b(1-x)^{c-b}(1-zx)^{-a}\frac{dx}{x(1-x)} \, .$$

Corresponds to $(x, 1-x, 1-zx) : \mathbb{P}^1 \setminus \{\infty, 0, 1, z^{-1}\} \longrightarrow \mathbb{G}_m^3$.

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Corresponds to $(x, 1-x, 1-zx) : \mathbb{P}^1 \setminus \{\infty, 0, 1, z^{-1}\} \longrightarrow \mathbb{G}_m^3$.

- Feynman integrals in dimensional regularization:

$$I_\Gamma(\varepsilon) = \int_{\sigma_\Gamma} \left(\frac{\Psi_\Gamma^{h+1}}{\Xi_\Gamma^h} \right)^{\varepsilon} \omega_\Gamma$$

Corresponds to $\mathbb{P}^{n-1} \setminus \{\Psi_\Gamma \Xi_\Gamma = 0\} \longrightarrow \mathbb{G}_m$.

(Not in this talk) Systems of finite difference equations

$$I_i(s+1) = \sum_{i=1}^{N} f_{i,j}(s)\, I_j(s) \quad \text{with } f_{i,j}(s) \in k(s).$$

- Example: $\mathrm{B}(s+1,t) = \dfrac{s}{s+t}\,\mathrm{B}(s,t)\ ,\ \mathrm{B}(s,t+1) = \dfrac{t}{s+t}\,\mathrm{B}(s,t).$
- Corresponds to a rank 1 “finite difference module” (Loeser–Sabbah).

(Not in this talk) Systems of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}z} I_i(s;z) = \sum_{j=1}^{N} g_{i,j}(s;z)\, I_j(s;z) \quad \text{with } g_{i,j}(s;z) \in k(s,z).$$

- Example: differential equation for $F(z) = {}_2F_1(a,b,c;z)$

$$z(1-z)\,F''(z) + (c-(a+b+1)z)\,F'(z) - ab\,F(z) = 0.$$

(Not in this talk) Values at $s \in \mathbb{Q}$

For $s \in \mathbb{Q}$, $I(s)$ is a period of a cyclic cover of X.

- Example: $\mathrm{B}(\frac{k}{d}, \frac{l}{d})$ is a period of an open Fermat curve $\{x^d + y^d = 1\}$.

(In this talk) Laurent expansion at $s = 0$

$$I(s) = \sum_{n \gg -\infty} \alpha_n s^n \quad \text{where the } \alpha_n \text{ are periods.}$$

- Example: $$\mathrm{B}(s,t) = \frac{s+t}{st}\left(1 - \sum_{m,n \geqslant 1} (-s)^m (-t)^n \zeta(\underbrace{1,\ldots,1}_{n-1}, m+1)\right).$$

What this talk is about...

- We are interested in the *motivic Galois theory* / *coaction* of the α_n.
- It is controlled by a *twisted cohomology group*.

The slogan

Galois theory of algebraic numbers *should* extend to a Galois theory for periods, where the Galois groups are *algebraic groups* over $\mathbb{Q}$.

- Periods arise as coefficients of the perfect pairing

$$\int : H_n^{\mathrm{B}}(X) \times H^n_{\mathrm{dR}}(X) \longrightarrow \mathbb{C} \ , \ (\sigma, \omega) \mapsto \int_\sigma \omega$$

  for algebraic varieties X, or pairs (X, Y), defined over $\mathbb{Q}$.
- A tannakian formalism for *motives* gives rise to a *motivic Galois group* that acts linearly on all $\mathrm{H}^n_{\mathrm{dR}}(X)$ and $\mathrm{H}^n_{\mathrm{dR}}(X, Y)$.
- This gives rise to a Galois theory for *periods*:

$$\text{“} \ g \, . \int_\sigma \omega := \int_\sigma g.\omega \ \text{”}$$

- Grothendieck’s *period conjecture* says that this formula is well-defined.
- Unconditional: Galois theory for *motivic periods*.
- Computable: Galois *coaction*.

The key example: the beta function

$$\mathrm{B}(s,t) = \frac{s+t}{st} \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) \left(s^n + t^n - (s+t)^n\right) \right).$$

- Galois theory for zeta values: for $g \in G_{\mathrm{dR}}$,

$$g \,.\, \zeta(n) = \zeta(n) + a_g^{(n)} \quad \text{with} \quad a_g^{(n)} \in \mathbb{Q}.$$

- We get a Galois theory for the beta function:

$$g \,.\, \mathrm{B}(s,t) = A_g(s,t)\, \mathrm{B}(s,t) \quad \text{with} \quad A_g(s,t) \in \mathbb{Q}((s,t))^{\times}.$$

- $\mathrm{B}(s,t)$ corresponds to a rank 1 representation of the motivic Galois group, defined over $\mathbb{Q}((s,t))$.

The main theorem

Theorem (Brown–D.–Fresán–Tapušković)

The motivic Galois group acts on Taylor expansions of algebraic Mellin transforms via power series, i.e., for g in the motivic Galois group G:

$$g.\int_\sigma f^s\omega = \sum_{i=1}^{N} A_g^{(i)}(s)\int_\sigma f^s\omega_i$$

where the $A_g^{(i)}(s)$ are in $k(\!(s)\!)$.

- This is a *finite* formula which computes the Galois theory of *infinitely many* periods.

Proof of concept

- A rank 2 example:

$$I(a;s) = \frac{1}{s}\left({}_2F_1(s,1,s+1;a) - 1\right) = \int_0^1 x^s \frac{a\,dx}{1-ax} = \sum_{n=0}^{\infty}(-s)^n\,\mathrm{Li}_{n+1}(a).$$

Galois theory (Deligne–Beilinson, Goncharov):

$$g.I(a;s) = A_g(a;s)\,I(a;s) + B_g(a;s) \quad \text{with} \quad A_g(a;s), B_g(a;s) \in \mathbb{Q}((s)).$$

- A family of examples (Brown-D. 2022): Lauricella hypergeometric functions

$$\int_0^{\sigma_i} x^{s_0}(1-x\sigma_1^{-1})^{s_1}\cdots(1-x\sigma_n^{-1})^{s_n}\frac{dx}{x-\sigma_j}.$$

Twisted cohomology

Twisted cohomology

X an (affine, smooth) algebraic variety over $\mathbb{C}$, $f : X \to \mathbb{C}^*$.

$$\mathrm{H}^\bullet(X, f) := \mathrm{H}^\bullet(X, f^*(t^s)).$$

- Fix $s \in \mathbb{C}$.
- de Rham: $\mathrm{H}^i_{\mathrm{dR}}(X, f) := \mathrm{H}^i(X, (\Omega^\bullet_X, \nabla_s))$ where
$$\nabla_s : \omega \mapsto d\omega + s\frac{df}{f} \wedge \omega \quad \text{(so that } d(f^s\omega) = f^s\nabla_s(\omega)\text{)}.$$
- Betti: $\mathrm{H}^{\mathrm{B}}_i(X, f) := \mathrm{H}^{\mathrm{sing}}_i(X, \mathcal{L}_s)$ where
$$\mathcal{L}_s = \mathbb{C} f^s \quad \text{(monodromy } e^{2\pi i s}\text{)}.$$
- Algebraic Mellin transforms arise as coefficients of the perfect pairing
$$\int : \mathrm{H}^{\mathrm{B}}_i(X, f) \times \mathrm{H}^i_{\mathrm{dR}}(X, f) \longrightarrow \mathbb{C} \ , \ (\sigma, \omega) \mapsto \int_\sigma f^s\omega.$$
- Easy to compute for *generic* values of $s \in \mathbb{C}$. Typical behavior:
$$\begin{cases} \mathrm{H}^i(X, f) = 0 \quad \text{for} \quad i \neq n := \dim(X); \\ \dim \mathrm{H}^n(X, f) = (-1)^n \chi(X). \end{cases}$$

Twisted cohomology, 2

Is twisted cohomology motivic?

- $\mathrm{H}^\bullet(X,f)$ *is not motivic* if $s \notin \mathbb{Q}$.
- A *formal generic* version of $\mathrm{H}^\bullet(X,f)$ *is motivic.*

- de Rham: a finite dimensional vector space over $k((s))$,

$$\mathrm{M}^i_{\mathrm{dR}}(X,f) := \mathrm{H}^i(X, (\Omega^\bullet_X((s)), \nabla)),$$

  where $\nabla : \omega \mapsto d\omega + s\frac{df}{f} \wedge \omega$.
- Betti: a finite dimensional vector space over $\mathbb{Q}((\log\mu))$,

$$\mathrm{M}^{\mathrm{B}}_i(X,f) := \mathrm{H}^{\mathrm{sing}}_i(X, \mathcal{L}),$$

  where $\mathcal{L}$ is the rank 1 local system of vector spaces over $\mathbb{Q}((\log\mu))$

$$\pi_1(X(\mathbb{C})) \xrightarrow{f_*} \pi_1(\mathbb{C}^*) = \mathbb{Z} \xrightarrow{\mu} \mathbb{Q}((\log\mu))^\times$$

- Perfect pairing valued in $\mathbb{C}((s))$, with $\mu \leftrightarrow e^{2\pi i s}$, giving rise to Laurent expansions of algebraic Mellin transforms.

Why is twisted cohomology motivic?

$$\begin{aligned}\mathrm{M}^i_{\mathrm{dR}}(X,f) &:= \mathrm{H}^i(X,(\Omega^\bullet_X((s)),\nabla))\\ &\simeq \Big(\varprojlim_n \underbrace{\mathrm{H}^i(X,(\Omega^\bullet_X[s]/(s^{n+1}),\nabla))}_{=:\,M_{n,\mathrm{dR}}}\Big)\otimes_{k[[s]]} k((s)).\end{aligned}$$

- Analogy with étale ℓ-adic cohomology:

$$\mathrm{H}^\bullet_{\text{ét}}(X;\mathbb{Q}_\ell) := \Big(\varprojlim_n \mathrm{H}^\bullet_{\text{ét}}(X;\mathbb{Z}/\ell^{n+1}\mathbb{Z})\Big)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell.$$

Each $M_{n,\mathrm{dR}}$ is motivic

- Comes from the *motivic fundamental group* of $\mathbb{G}_m$ (Hain, Deligne).
- The $k[s]/(s^{n+1})$-module structure is motivic, where $s \leftrightarrow \mathrm{H}_1(\mathbb{G}_m)$.
- Tannakian category of “local Mellin motives”

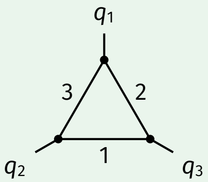
$$\mathrm{M}(X,f) = (\cdots \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0).$$

Application to Feynman integrals

- Γ a connected graph with n edges and first Betti number h.
- Graph polynomials Ψ_Γ, Ξ_Γ, homogeneous in n variables.
- Feynman integral

$$I_\Gamma = \int_{\mathbb{P}^{n-1}(\mathbb{R}_+)} \frac{\Psi_\Gamma^{n-(h+1)D/2}}{\Xi_\Gamma^{n-hD/2}}\,\Omega.$$

Example: the massless triangle graph ($D = 4$)



$$I_\Gamma = \iint_{(0,\infty)^2} \frac{dx\,dy}{(x+y+1)(q_1^2 x + q_2^2 y + q_3^2 xy)}$$

Problem: Feynman integrals *do not always converge*!

A wild idea

Work in space-time dimension

$$D = D_0 - 2\varepsilon$$

and consider the Laurent expansion near $\varepsilon = 0$.

Example: the massless triangle graph ($D_0 = 4$)

$$I_\Gamma(\varepsilon) = \iint_{(0,\infty)^2} \left(\frac{(x+y+1)^2}{q_1^2 x + q_2^2 y + q_3^2 xy} \right)^{\varepsilon} \frac{dxdy}{(x+y+1)(q_1^2 x + q_2^2 y + q_3^2 xy)}$$

- This is an algebraic Mellin transform for

$$f = \frac{\Psi_\Gamma^{h+1}}{\Xi_\Gamma^h} : X = \mathbb{P}^{n-1} \setminus \{\Psi_\Gamma \Xi_\Gamma = 0\} \longrightarrow \mathbb{G}_m.$$

- Corresponding geometry: $(X, \bigcup_i \{x_i = 0\}, f)$.

Theorem (Brown–D.–Fresán–Tapušković)

The space of Laurent expansions of Feynman integrals in dimensional regularization is closed under the action of the motivic Galois group:

$$g.I_\Gamma(\varepsilon) = \sum_{i=1}^{N} A_g^{(i)}(\varepsilon)\, I_{\Gamma_i}(\varepsilon) \quad \text{with} \quad A_g^{(i)}(\varepsilon) \in \overline{\mathbb{Q}}((\varepsilon)).$$

- Conjectured and checked by Abreu–Britto–Duhr–Gardi–Matthew.
- Still difficult to make explicit.
- Should lead to arithmetic constraints on Feynman integrals.