4-12/14

2022年4月10日 9:46

Reference

- [S14] Sorcar, "Teichmüller space of negatively curved metrics on Gromov-Thurston mfds is not contractible", 2014.
- [FS17] Farrell, Sorcar, "Teichmüller space of neg. curved metrics on complex hyp. mfd is not contractible", 2017.
- [GT87] Gromov, Thurston, "Pinching constants for hyp. mfds.", 1987.
- [A00] Ardanza, "Exotic smooth structures on non-locally symmetric negatively curved mfds". PhD Thesis. Binghamton university, 2000.
- [BP] Benedetti-Petronio, Lectures on hyperbolic geometry.

Relate Lemma 3.9′ with the following th.

<u>Th 3.3</u> ([FO09]) Let M be a closed real hyp. mfd of dim $n \geq 6$. Assume that $\Theta_{n+1} \neq 0$. Then $\exists$ finite cover N of M s.t.

$$\pi_1 T^{<0}(N) \xrightarrow{F_*} \pi_1 T(N) \quad \text{is nontrivial}$$

group of homotopy spheres.

We have sketched the pf of <u>Th3.3</u> when $n=6$. Recall the strategy is to find $f \in \mathrm{Diff}_0(N)$ s.t.

$$\pi_0 \mathrm{Diff}_0(N) \longrightarrow \pi_0 R^{<0}(N)$$
$$[f] \neq 0 \longmapsto [f_* g] = [g]$$

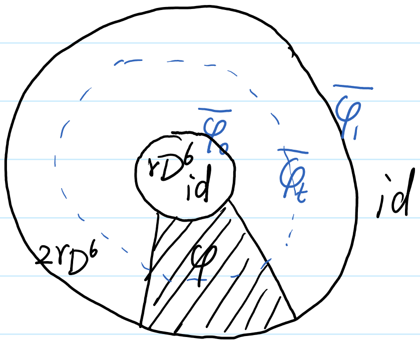
the given hyp. metric on N.

Construction of f: begin with $\varphi \in \mathrm{Diff}(D^6, \partial)$ s.t.

$$\pi_0 \mathrm{Diff}(D^6, \partial) = \Theta_7$$

and $\Omega\mathrm{Diff}(D^5,\partial) \xrightarrow[\alpha]{[\varphi]\neq 0} \mathrm{Diff}(D^6,\partial)$

$\exists\, \cdot \longmapsto \varphi$



$\subset N^6 \circlearrowleft f$

$$\Omega\mathrm{Diff}(D^5,\partial) \xrightarrow{\alpha} \mathrm{Diff}(D^6,\partial) \xrightarrow{\iota} \mathrm{Diff}_0(N)$$

$$\exists\, \cdot \longmapsto \varphi \longmapsto f$$

$$f \in \mathrm{Im}(\iota\circ\alpha) \xRightarrow[\text{N has a geod. ball of radius} \gg 0]{\text{tapering process}} [f_* g] = [g] \in \pi_0 R^{<0}(N)$$

pf Lemma 3.9:

Let $\mathrm{Diff}_0(S^{n-1}\times[1,2],\partial) \triangleq \{ f\in \mathrm{Diff}(S^{n-1}\times[1,2]) \mid$

$f|_{\text{near } S^{n-1}\times\{1,2\}} = id$ and $f \sim id$ rel near $S^{n-1}\times\{1,2\}\}$

with C^∞-top; forms a top. grp.

generalized Gromoll's map.

$$\Omega \mathrm{Diff}_0(S^{n-1}) \xrightarrow{\beta} \mathrm{Diff}_0(S^{n-1}\times[1,2],\partial)$$

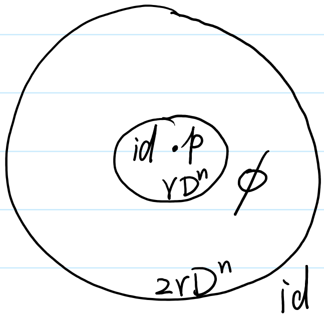
N^n: real hyp. mfd with geodesic ball B of radius $2r$ at $p\in N$.

$\Longrightarrow$ Identify $B\setminus p$ with $S^{n-1}\times(0,2r]$

Identify $S^{n-1}\times[r,2r]$ with $S^{n-1}\times[1,2]$.

$$\rightsquigarrow \Lambda(N,p,r): \mathrm{Diff}_0(S^{n-1}\times[1,2],\partial) \longrightarrow \mathrm{Diff}_0(N)$$
$$\phi \longmapsto \Lambda(N,p,r)(\phi)$$

$\to$ extend by id on $N\setminus S^{n-1}\times[r,2r]$



$\subset N \circlearrowleft \Lambda(N,p,r)(\phi)$

<u>Lemma 3.12</u> ([FO09, Theorem 2]) Given a compact subset $K\subset \mathrm{im}\,\beta$, $\exists\, r>0$ s.t. following holds:

let (N,g) be a closed, real hyp. mfd and let $p\in N$ with injectivity radius at $p > 3r$, then the map

$$K\subset \mathrm{im}\beta \subset \mathrm{Diff}_0(S^{n-1}\times[1,2],\partial) \xrightarrow{\Lambda(N,p,r)} \mathrm{Diff}_0(N) \longrightarrow \mathcal{R}^{<0}(N)$$

is null-homotopic.

Now proceed to prove Lemma 3.9 by assuming Lemma 3.12 is true.

Embed $D^{n-1}\hookrightarrow S^{n-1} \rightsquigarrow \mathrm{Diff}(D^{n-1},\partial)\longrightarrow \mathrm{Diff}_0(S^{n-1})$

$$\rightsquigarrow D^n = D^{n-1}\times[1,2] \hookrightarrow S^{n-1}\times[1,2]$$

$$j: \mathrm{Diff}(D^n,\partial) \longrightarrow \mathrm{Diff}_0(S^{n-1}\times[1,2],\partial)$$

$$\Rightarrow \begin{array}{ccc} \Omega\mathrm{Diff}(D^{n-1},\partial) & \xrightarrow{\alpha_n} & \mathrm{Diff}(D^n,\partial) \\ \downarrow & \circlearrowleft & \downarrow j \\ \Omega\mathrm{Diff}(S^{n-1}) & \xrightarrow{\beta} & \mathrm{Diff}_0(S^{n-1}\times[1,2],\partial) \end{array}$$

$$\Omega \mathrm{Diff}_0(S^{n-1}) \xrightarrow{\beta} \mathrm{Diff}_0(S^{n-1}\times[1,2],\partial)$$

Let $f: S^i \to \Omega \mathrm{Diff}(D^{n-1},\partial)$ s.t. $[\alpha_n \circ f] = \bar{x} \in \pi_i \mathrm{Diff}(D^n,\partial)$

Take $K = \mathrm{Im}(j\circ\alpha_n\circ f) = \{(j\circ\alpha_n\circ f)(u) \mid u\in S^i\}$.

$\Rightarrow$ K is compact and $K \subset \mathrm{im}\beta$.

Lemma 3.12 and let $r>0$ be in 3.12.

N: closed, real, hyp. mfd with injectivity radius $>3r$ at p.

$$\xRightarrow{\text{Lemma 3.12}} K \xrightarrow{\Lambda(N,p,r)} \mathrm{Diff}_0(N) \xrightarrow{\sigma_g} \mathcal{R}^\circ(N)$$
$$f \longmapsto f_* g = (f^{-1})^* g.$$

Consider

$$\begin{array}{c} \pi_i \Omega \mathrm{Diff}(D^{n-1},\partial) \\ \| \\ \pi_{i+1}\mathrm{Diff}(D^{n-1},\partial) \end{array} \xrightarrow{\alpha_{n*}} \pi_i \mathrm{Diff}(D^n,\partial)$$

$$[f] \longmapsto \bar{x}$$

$$\downarrow j_*$$

$$\mathrm{Im}(j_*) \in \pi_i K \longrightarrow \pi_i \mathrm{Diff}_0(S^{n-1}\times[1,2],\partial)$$

$$\downarrow \Lambda(N,p,r)_*$$

$$\pi_i \mathrm{Diff}_0(N) \xrightarrow{\sigma_{g*}} \pi_i \mathcal{R}^\circ(N)$$

$\Rightarrow$ $(\sigma_g \circ \Lambda(N,p,r)\circ j)_* \bar{x} = 0$

$\Rightarrow$ $(\Lambda(N,p,r)\circ j)_* \bar{x} \in \mathrm{Im} F_*$.

$\square$

We have seen

$$\mathcal{T}^\circ(M) \xrightarrow{F_*} \mathcal{T}(M)$$

is (rationally-) homotopically nontrivial for many hyp. mfd M of high dim.

<u>Prob 3.13</u>: How about

$$F_*: \pi_i T^{<0}(M) \longrightarrow \pi_i T(M)$$

for a neg. curved mfd M which does not support a hyp. metric?

Sorcar studied Prob 3.13 for some Gromov–Thurston's mfds cf. [S14].

- $\not\cong$ locally symmetric space $\Rightarrow$ not support real hyp. metric
- but support neg. curved metric

<u>Th 3.14 ([S14])</u> For every $n=4k-2$ where $k \in \mathbb{Z}_{\geq 2}$, $\exists$ neg. curved mfd M^n s.t.

$M \not\cong$ locally symmetric space

and $F_*: \pi_1 T^{<0}(M) \longrightarrow \pi_1 T(M)$ is nontrivial.

($\Rightarrow \pi_1 T^{<0}(M) \neq 0 \Rightarrow T^{<0}(M) \not\simeq *$).

Farrell-Sorcar studied <u>Prob 3.13</u> for complex hyp. mfd [FS17].

<u>Th 3.15</u> ([FS17]) For every $n=4k-2$ where $k \in \mathbb{Z}_{\geq 2}$, $\exists$ complex hyp. mfd M^n s.t.

$$\pi_1 T^{<0}(M) \xrightarrow{F_*} \pi_1 T(M)$$

is nontrivial ($\Rightarrow \pi_1 T^{<0}(M) \neq 0 \Rightarrow T^{<0}(M) \not\simeq *$).

To show <u>Th 3.14</u>, first introduce Gromov–Thurton

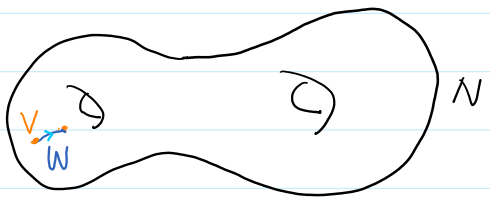
To show Th3.14, first introduce Gromov–Thurston mfds, cf. [GT87] [A00].

[GT87] construct these mfds by taking branched covers over a codimension two totally geodesic submfd of a closed real hyp. mfd.

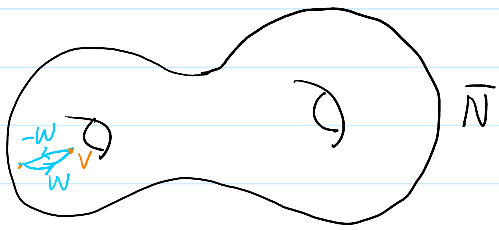
Let N^n be a closed, oriented, real hyp mfd s.t.

$$V^{n-2} \subset W^{n-1} \subset N$$

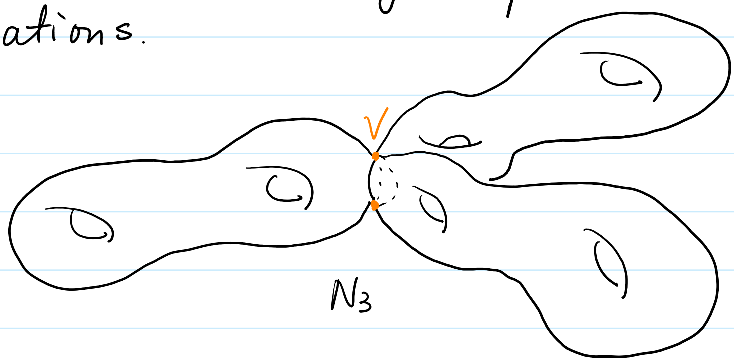
with $V = \partial W$, where V and W are oriented totally geodesic submfd of N.



Cut N along W to produce $\bar{N}$ with $\partial \bar{N} = W \cup (-W)$ → W with opposite orientation



For $s \in \mathbb{Z}_{>1}$, take s copies of $\bar{N}$ and paste them along their boundaries in a way compatible with their orientations.



$\rightsquigarrow N_s$, $\rho : N_s \longrightarrow N$ a branched cover of N along V.

Def Given a Riem. mfd N, a closed submfd $V \subset N$. The <u>normal injectivity radius of</u> $V \subset N$ is the greatest real number r for which the open r-neighborhood $U_r \subset N$ of V is diffeomorphic to the normal bundle of V in N, denoted by $\mathrm{Rad}^{\perp}(V)$

Th 3.16 ([GT87]) Given $R > 0$ and

N^n: closed, oriented, hyp. mfd.

V^{n-2}: totally geodesic submfd that bounds a connected oriented totally geodesic hypersurface $W \subset N$. with $\mathrm{Rad}^{\perp}(V) \geq R$

$\rho: N_s \longrightarrow N$ is a branch cover of N as described before.

$\Longrightarrow$ $\exists$ metric g_s on N_s with negative sectional curvature K s.t. $K \equiv -1$ outside the open R-neighborhood of the branching set V.

Now give examples satisfy the assumptions in Th 3.16.

Example ([GT87]) These examples are

$\mathbb{H}^n / \Gamma \longrightarrow$ discrete, cocompact, torsion-free subgrp of $\mathrm{Isom}(\mathbb{H}^n)$.

Consider <u>hyperboloid model for $\mathbb{H}^n$:</u> $\longrightarrow$ cf. [BP]

Given the quadratic form

$$\Phi_n(x) = x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2$$

$$I \triangleq \left\{ x \in \mathbb{R}^{n+1} \,\middle|\, \Phi_n(x) = -1,\ x_{n+1} > 0 \right\}$$

Since -1 is the regular value of Φ_n $\Rightarrow$ I is a differentiable oriented hypersurface in $\mathbb{R}^{n+1}$.

$T_x \mathbb{I} = \{ v \in T_x\mathbb{R}^{n+1} \mid d\Phi_n(v) = 0 \}$

$v = \sum_{i=1}^{n+1} y_i \frac{\partial}{\partial x_i}$. $d\Phi_n(v) = 0 \iff x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1} = 0$

$\Rightarrow T_x \mathbb{I} = \{ y \in \mathbb{R}^{n+1} \mid B(x,y) = 0 \} = \{x\}^{\perp}$

where $B : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$

$(u, v) \longmapsto u_1v_1 + u_2v_2 + \cdots + u_nv_n - u_{n+1}v_{n+1}$

$B|_{T_x\mathbb{I}}$ is pos-def. (Since $B(x,x) = -1$)

$\Downarrow$

a Riem. metric on $\mathbb{I}$.

$\Downarrow$

$\mathbb{I}$ is the hyperboloid model for $\mathbb{H}^n$.

$O(\mathbb{R}^{n+1}, B) = \{ A \in GL_{n+1}(\mathbb{R}) \mid B(Av, Au) = B(v,u)$ for all $v, u \in \mathbb{R}^{n+1} \}$

Fact: $\mathrm{Isom}(\mathbb{H}^n) = \{ A \in O(\mathbb{R}^{n+1}, B) \mid A(\mathbb{I}) = \mathbb{I} \}$

the group of isometries of $\mathbb{H}^n$.

cf. [BP, ThA.2.4].

Let $\phi_n(x) \triangleq x_1^2 + \cdots + x_n^2 - \sqrt{2}\, x_{n+1}^2$

$\mathbb{I}(\sqrt{2}) = \{ x \in \mathbb{R}^{n+1} \mid \phi_n(x) = -1,\ x_{n+1} > 0 \}$

Let $B_{\sqrt{2}} : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$

$(u, v) \longmapsto u_1v_1 + \cdots + u_nv_n - \sqrt{2}\, u_{n+1}v_{n+1}$

$G(\sqrt{2}) \triangleq \{ A \in O(\mathbb{R}^{n+1}, B_{\sqrt{2}}) \mid A(\mathbb{I}(\sqrt{2})) = \mathbb{I}(\sqrt{2}) \}$.

We identify $G(\sqrt{2})$ with $\mathrm{Isom}(\mathbb{H}^n)$ as follows: