2021-10-04 Kähler geometry

复习 ①

$k=0,-1$ case $\longleftrightarrow$ $c_1(M)=0$, $c_1(M)<0$

<u>Yau</u> $\exists$ KE metrics

Application

Miyaoka-Yau inequality

Miyaoka (1977)

M surface of general type

(Kodaira dim $=2$)

$\Rightarrow$ $c_1^2 \le 3c_2$

Yau (1977)

$c_1(M)<0 \Rightarrow (-1)^m \frac{2(m+1)}{m} c_1^{m-2} c_2$

$\geq (-1)^m c_1^2$

Equality $\Longleftrightarrow$ universal cover is $B^m=\{z\in\mathbb{C}^m \mid |z|<1\}$

<u>Rem</u> $c_1<0 \Rightarrow$ general type

~~Is converse True ?~~

~~Yes for $k<0$. Yau, Aubin 1976~~

~~Yes for $k=0$ Yau~~

~~Calabi-Yau mfd~~

~~the metric is called the Calabi-Yau metric~~

~~No in general for $k>0$. Matsushima~~

~~K-stability is necessary and sufficient.~~

How is the proof for $k<0$ and $k=0$.

ω fixed Kähler m. $[\omega] \in H^2_{DR}(M : \mathbb{R})$

Kähler class

$\sqrt{-1}\, g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$

Ω = the set of all Kähler forms cohomologous to ω.

$\forall \omega' \in \Omega \qquad \omega' - \omega = \sqrt{-1}\,\partial\bar{\partial}\varphi \qquad \exists \varphi \in C^\infty(M)$.

by the $\partial\bar{\partial}$-lemma.

③ ~~(4)~~

Suppose $c_1(M) = 0$.

$$-\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log \det(g_{i\bar j}) \quad (*)$$

$$c_1(g) = \frac{\sqrt{-1}}{2\pi} R_{i\bar j}\, dz^i \wedge d\bar z^j = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial F$$

for $^{\exists} F \in C^\infty(M)$.

If $^{\exists} g'$ s.t. $c_1(g') = 0$ with $\omega' \in \Omega$.

then $\cdot$ $\omega' = \omega + \sqrt{-1}\,\partial\bar\partial\varphi$.

i.e. $g'_{i\bar j} = g_{i\bar j} + \dfrac{\partial^2 \varphi}{\partial z^i \partial \bar z^j}$ $= \varphi_{i\bar j}$

$= g_{i\bar j} + \varphi_{i\bar j}$

$\cdot$ $-\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log\det(g'_{i\bar j}) = 0$. $\quad (**)$

$(*) - (**)$

$$\partial\bar\partial \log \frac{\det(g_{i\bar j} + \varphi_{i\bar j})}{\det(g_{i\bar j})} = \partial\bar\partial F$$

$$\therefore \log \frac{\det(g_{i\bar j} + \varphi_{i\bar j})}{\det(g_{i\bar j})} = \overbrace{F + \text{const}}^{\to F}$$

$$\left(\partial\bar\partial f = 0 \Rightarrow \underset{g^{i\bar j}\nabla_i\nabla_{\bar j} f}{\underset{\shortparallel}{\Delta f}} = 0 \Rightarrow f = \text{const} \right)$$

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{F} \quad \text{———} (1).$$

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② Given g and F on a compact Kähler mfd M, can we solve this equation (1) ?

Monge-Ampère equation.

= A PDE involving the determinant of a function.

Theorem (Yau 1976):

(1) is solvable on <u>any compact Kähler manifold.</u>

The case $c_1(M) < 0$ or $c_1(M) > 0$.

We may consider $-c_1(M)$ or $c_1(M)$ to be the Kähler class. So $\Omega = k c_1(M)$, $k = -1$ or 1.

Let $\omega \in k c_1(M)$ be a Kähler form.

Since $\mathrm{Ric}(\omega) \in c_1(M)$, $\exists F \in C^{\infty}(M)$ s.t.

$$\mathrm{Ric}(\omega) - k\omega = i\partial\bar\partial F. \quad \text{———} \quad (*)$$

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If $\exists\, \omega' \in k c_1(M)$ s.t.

$$\mathrm{Ric}(\omega') = k\omega' \qquad \text{(K-E metric)} \quad (**)$$

Write $\omega = \sqrt{-1}\, g_{i\bar j}\, dz^i \wedge d\bar z^j$

$\omega' = \omega + i\partial\bar\partial\varphi$ so $g'_{i\bar j} = g_{i\bar j} + \varphi_{i\bar j}$.

$(*) - (**)$ gives

$$\frac{\det(g_{i\bar j} + \varphi_{i\bar j})}{\det(g_{i\bar j})} = e^{-k\varphi + \underbrace{F + c}_{F}}$$

We may replace $F + c$ by F.

Theorem (Aubin, Yau, $k = -1$, 1976)

$$\frac{\det(g_{i\bar j} + \varphi_{i\bar j})}{\det(g_{i\bar j})} = e^{\varphi + F} \text{ is solvable}$$

on any compact Kähler mfd.

This $k = -1$ case is much easier than the case of $k = 0$.

By Matsushima, $k = 1$ case,

$$\frac{\det(g_{i\bar j} + \varphi_{i\bar j})}{\det(g_{i\bar j})} = e^{-\varphi + F} \quad \text{———} \quad (*)$$

does not have a solution in general.

Continuity method by Yau

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{-k\varphi + tF} \quad (*_t)$$

$$0 \leq t \leq 1.$$

$S := \{t \in [0,1] \mid *_t \text{ has a solution.}$

If we can show S is nonempty, open, and closed in $[0,1]$, then $S = [0,1]$, and $(*_1)$ for $t=1$ has a solution.

(1) $0 \in S$, indeed $\varphi = 0$ is a solution. So $S \neq \phi$.

(2) For $k = -1, 0$, one can show using the implicit function theorem for

$$\Delta + k : C^{l+\alpha}(M) \to C^{l-2+\alpha}$$

S is open.

(3) For $k = -1, 0$, one can show S is closed.

For $k = 1$, (2) and (3) are difficult. (impossible in general).

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To show the closedness of S enough to show the following.

For a solution φ_t for $(*_t)$, $t \in S$,

$\{\varphi_t\}_{t\in S}$ is bounded in $C^3(M)$, i.e.

$$|\varphi|, |\nabla\varphi|, |\nabla\nabla\varphi|, |\nabla\nabla\nabla\varphi| \leq C \quad \text{indep of } t$$

Then by Arzela-Ascoli, a subsequence converges to a solution.

Yau proved the above estimates for $k = -1, 0$.

Even for $k = 1$, Yau's work shows that it is sufficient to show

$$|\varphi_t| \leq C \quad \text{indep } t \quad (C^0\text{-estimate})$$

Namely, C^0-estimate implies C^3-estimate.