- lemma: $\delta_m(L) = \inf_F \dfrac{A_X(F)}{S_m(L,F)}$ $\qquad F$ prime divisor over X

$S_m(L,F) = \frac{1}{m d_m} \sum_{j\geq 1} \dim H^0(X, mL - jF)$

pf: Claim: $S_m(L,F) = \max\{ \mathrm{ord}_F(D) : D \text{ } m \text{ basis divisor of } L \}$.

max is attained exactly by those D s.t. the basis defining D is compatible with the filtration given by F:

$$H^0(X, mL) \supseteq H^0(X, mL - F) \supseteq H^0(X, mL-2F) \supseteq \cdots \supseteq \{0\}.$$

$H^0(X, mL - jL) \subseteq H^0(X, mL - (j-1)L)$

Indeed, for $\forall$ basis $\{s_i\}_{i=1}^{d_m}$ of $H^0(X, mL)$ let

$$a_j := \#\{ s_i : \mathrm{ord}_F(s_i) \geq j \}. \Rightarrow a_j \leq \dim H^0(X, mL - jF)$$

Then for $D := \frac{1}{m d_m} \sum_{i=1}^{d_m} (s_i = 0)$, we have

$$\mathrm{ord}_F(D) = \frac{1}{m d_m} \sum_{i=1}^{d_m} \mathrm{ord}_F(s_i) = \frac{1}{m d_m} \sum_{j\geq 0} j(a_j - a_{j+1}).$$

$$= \frac{1}{m d_m} \sum_{j \geq 1} a_j \leq \frac{1}{m d_m} \sum_{j\geq 1} \dim H^0(X, mL - jF).$$

Observe: equality happens iff $\{s_i\}$ is compatible w/ the filtration

H-ONB

$$\delta_m(L) = \inf_{D \text{ m basis}} \mathrm{lct}(X,D) = \inf_{D \text{ m basis}} \inf_F \frac{A_X(F)}{\mathrm{ord}_F(D)} \geq \inf_{D \text{ m basis}} \inf_F \frac{A_X(F)}{S_m(L,F)}.$$

$$\delta_m(L) \geq \inf_F \frac{A_X(F)}{S_m(L,F)}.$$

$$\inf_F \frac{A_X(F)}{S_m(L,F)} = \inf_F \frac{A_X(F)}{\mathrm{ord}_F(D_F)} \geq \inf_F \mathrm{lct}(X, D_F) \geq \inf_{D_m \text{ basis}} \mathrm{lct}(X,D) = \delta_m(L).$$

D_F: compatible basis divisor of F.

H-ONB.

$\Rightarrow$ we conclude. $\square$

- **Cor:** Fix a Hermitian inner product H on $H^0(X, mL)$.
  Then $\delta_m(L) = \inf\{\mathrm{lct}(X,D) \mid D \text{ is induced by an } H\text{-orthonormal basis}\}$.

- <u>Rmk</u>: D m basis divisor induced by $\{s_i\}_{i=1}^{d_m}$ then

$$\mathrm{lct}(X,D) = \sup\left\{\lambda > 0 \;\middle|\; \int_X \frac{dv}{\left(\prod |s_i|^2_{h^m}\right)^{\frac{\lambda}{m d_m}}} < \infty\right\}.$$

- **Cor**: $\delta_m(L) = \sup\left\{ \lambda > 0 \;\middle|\; \int_X \frac{dV}{(\prod |S_i|^2_{h^m})^{\frac{\lambda}{m}}} dm < \infty \text{ for all } H\text{-ONB } \{S_i\} \right\}$.

- **Prop**: $\delta_m(L) = \sup\left\{ \lambda > 0 \;\middle|\; \sup_{\substack{H\text{-ONB} \\ \{S_i\}}} \int_X \frac{dV}{(\prod |S_i|^2_{h^m})^{\frac{\lambda}{m}}} dm < \infty \right\}$.

Pf: Goal: $\delta_m(L) \leq$ RHS.

We argue by contradiction. Say $\delta_m(L) >$ RHS.

So we can find some $\lambda < \delta_m(L)$ but $\exists$ a sequence of H-ONB $\{S_i^{(k)}\}$, $k \to \infty$ s.t.

$$\int_X \frac{dV}{(\prod |S_i^{(k)}|^2_{h^m})^{\frac{\lambda}{m}}} dm \longrightarrow \infty.$$

Up to a subsequence, $\{S_i^{(k)}\} \to \{S_i^{(\infty)}\}$. H-ONB.

$$\int_X \frac{dV}{(\prod |S_i^{(\infty)}|^2_{h^m})^{\frac{\lambda}{m}}} dm < \infty.$$

This is impossible! (This follows from Demailly–Kollár's lower semi-continuity of lct).

$\updownarrow$

Openness conjecture. Guan–Zhou.

$\square$

- Prop: $\delta_m^A(L) \geq \delta_m(L)$.

Pf: $\delta_m^A(L) = \sup\left\{ \lambda > 0 \;\middle|\; \sup_{\varphi \in B_m} \int_X e^{-\lambda(\varphi - E_m(\varphi))} dV < \infty \right\}$

$$= \sup\left\{ \lambda > 0 \;\middle|\; \sup_{\substack{H\text{-ONB } \{\sigma_i\} \\ \lambda_i > 0}} \int_X \frac{\left(\prod_{i=1}^{d_m} \lambda_i^2\right)^{\frac{\lambda}{m d_m}}}{\left(\sum_{i=1}^{d_m} \lambda_i^2 |\sigma_i|_{h^m}^2\right)^{\lambda/m}} dV < \infty \right\}$$

$$\geq \left(\prod_{i=1}^{d_m} \lambda_i^2 |\sigma_i|_{h^m}^2\right)^{1/d_m}$$

$$\geq \sup\left\{ \lambda > 0 \;\middle|\; \sup_{H\text{-ONB } \{\sigma_i\}} \int_X \frac{dV}{\left(\prod_{i=1}^{d_m} |\sigma_i|_{h^m}^2\right)^{\lambda/m d_m}} < \infty \right\} = \delta_m(L).$$

$\square$

Prop: $\delta_m^A(L) \leq \delta_m(L)$.

Pf: We argue by contradiction.

Assume $\delta_m^A(L) > \delta_m(L)$. Pick $\lambda \in (\delta_m(L), \delta_m^A(L))$.

Recall $\delta_m(L) = \inf_F \frac{A_X(F)}{S_m(F)}$, so we can find some prime divisor F/X

$F \subseteq Y \xrightarrow{\pi} X$ and it satisfies

$$\frac{A_X(F)}{S_m(F)} < \lambda \Rightarrow \exists \varepsilon > 0 \text{ s.t. } \varepsilon + A_X(F) \leq \lambda S_m(F).$$

We look at the filtration

$$H(\cdot,\cdot) = \int_X h^m(\cdot,\cdot)\omega^n.$$

$$H^0(X, mL) \supseteq H^0(X, mL - F) \supseteq \cdots \supseteq \{0\}.$$

Pick an H-ONB $\{s_i\}$ compatible with this filtration.

let $\varphi_t := \frac{1}{m} \log \sum_{i=1}^{d_m} e^{\mathrm{ord}_F(s_i)t} |s_i|^2_{h^m}$, $t \geq 0$.

This is a family of Bergman potentials in $\mathcal{B}_m$ given by $\{e^{\frac{\mathrm{ord}_F(s_i)t}{2}} s_i\}$.

"this is a geodesic ray of $\mathcal{B}_m = GL(d_m)/U(d_m)$."

$$E_m(\varphi_t) = \frac{1}{m d_m} \log \prod_{i=1}^{d_m} e^{\mathrm{ord}_F(s_i) t} = \frac{1}{m d_m} \sum_{i=1}^{d_m} \mathrm{ord}_F(s_i) t$$

$$= S_m(F)\, t.$$

Put

$$f(t) := \int_X e^{-\lambda(\varphi_t - E_m(\varphi_t))} dV = \int_X \frac{e^{\lambda S_m(F) t}}{\left(\sum_{i=1}^{d_m} e^{\mathrm{ord}_F(s_i)} |s_i|^2_{h^m}\right)^{\lambda/m}} dV$$

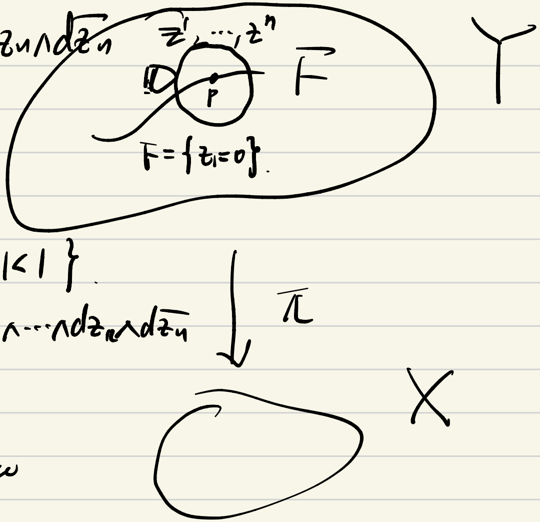
$$\geq e^{\varepsilon t} \int_X \frac{e^{A_X(F) t}}{\left(\sum_{i=1}^{d_m} e^{\mathrm{ord}_F(s_i) t} |s_i|^2_{h^m}\right)^{\lambda/m}} dV$$

It suffices to show that

$$\int_X \frac{e^{A_X(F) t}}{\left(\sum_{i=1}^{d_m} e^{\mathrm{ord}_F(s_i) t} |s_i|^2_{h^m}\right)^{\lambda/m}} dV \geq C > 0 \quad \forall\, t \geq 0.$$

$$\pi^* dV \sim |z_1|^{2A_X(F)-2} (\sqrt{-1})^n dz_1\wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

$$s_i = z_1^{\mathrm{ord}_F(s_i)} f_i \qquad h \leq C. \quad f_i \leq C$$



Locally
$$\int_{\mathbb{D}} \frac{e^{tA_X(F)} |z_1|^{2A_X(F)-2}}{\left(\sum_{i=1}^{d_m} e^{t\,\mathrm{ord}_F(s_i)} |z_1|^{2\,\mathrm{ord}_F(s_i)}\right)} (\sqrt{-1})^n dz_1\wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

$\mathbb{D} = \{|z_i| < 1\}$.

e.x. Show this is uniform bounded from below by $c > 0$ for all $t \geq 0$.

$\square$.

$$\delta^A(L) = \sup\left\{\lambda > 0 \,\middle|\, \sup_{\varphi \in \mathcal{H}_\omega} \int e^{-\lambda(\varphi - E(\varphi))} dV < \infty\right\}$$

$$\delta_m^A(L) = \sup\left\{\lambda > 0 \,\middle|\, \sup_{\varphi \in \mathcal{B}_m} \int_X e^{-\lambda(\varphi - E_m(\varphi))} dV < \infty\right\}.$$

$E_m(\varphi_m) \longrightarrow E(\varphi)$. for $\forall \varphi \in \mathcal{H}$. $\exists\, \varphi_m \in \mathcal{B}_m$ s.t. $\varphi_m \to \varphi$.

pf of $\delta^A(L) \geq \limsup_{m \to \infty} \delta_m^A(L)$

For $\forall\, \lambda < \limsup \delta_m^A(L)$, need to find $C > 0$ s.t.

$$\int_X e^{-\lambda(\varphi - E(\varphi))} dV \leq C \quad \text{for } \forall\, \varphi \in \mathcal{H}_\omega, \ \sup\varphi = 0.$$

Let $\varepsilon > 0$, $m \geq m_0(X, L, \varepsilon)$, to be fixed later,

Then

$$\int_X e^{-\lambda\varphi + \lambda E(\varphi)} dV \leq \int_X e^{-\lambda\varphi + \lambda E_m(((1-\varepsilon)\varphi)_m)} dV = \int_X e^{-\lambda(1-\varepsilon)\varphi - \lambda\varepsilon\varphi + \lambda E_m(((1-\varepsilon)\varphi)_m)} dV$$

$$= \int_X e^{\lambda((1-\varepsilon)\varphi)_m - \lambda(1-\varepsilon)\varphi - \lambda((1-\varepsilon)\varphi)_m + \lambda E_m(((1-\varepsilon)\varphi)_m) - \lambda\varepsilon\varphi} dV$$

$$= \int_X e^{\lambda(((1-\varepsilon)\varphi)_m - (1-\varepsilon)\varphi)} \cdot e^{-\lambda(((1-\varepsilon)\varphi)_m - E_m(((1-\varepsilon)\varphi)_m))} \cdot e^{-\lambda\varepsilon\varphi} dV.$$

Hölder

$$\leq \left(\int_X e^{\lambda p(((1-\varepsilon)\varphi)_m - (1-\varepsilon)\varphi)} dV\right)^{1/p} \left(\int_X e^{-\lambda q(((1-\varepsilon)\varphi)_m - E_m(((1-\varepsilon)\varphi)_m))} dV\right)^{1/q} \left(\int_X e^{-\lambda\varepsilon r\varphi} dV\right)^{1/r}$$

$p, q, r > 0$

$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$

$$\begin{cases} \lambda p = m \\ \lambda\varepsilon r = \alpha \in (0, \alpha(L)) \end{cases} \Longrightarrow \begin{cases} p = \frac{m}{\lambda} \\ r = \frac{\alpha}{\lambda\varepsilon} \\ q = \frac{1}{1 - \frac{\lambda}{m} - \frac{\lambda\varepsilon}{\alpha}} \end{cases}$$

$$\int_X e^{-\lambda(\varphi - E(\varphi))} dV \leq (d_m)^{\frac{\lambda}{m}} \underbrace{\left(\int_X e^{-\frac{\lambda}{1-\frac{\lambda}{m}-\frac{\lambda\varepsilon}{\alpha}}(((1-\varepsilon)\varphi)_m - E_m(((1-\varepsilon)\varphi)_m))} dV\right)^{(1-\frac{\lambda}{m}-\frac{\lambda\varepsilon}{\alpha})}}_{\leq C_1} \underbrace{\left(\int_X e^{-\alpha\varphi} dV\right)^{\frac{\lambda\varepsilon}{\alpha}}}_{\leq C_2}$$

Finally, choose ε small enough and $m \geq m_0(X, L, \varepsilon)$ large enough

s.t. $\dfrac{\lambda}{1-\frac{\lambda}{m}-\frac{\lambda\varepsilon}{\alpha}} < \delta_m^A(L)$.

So $\displaystyle\int_X e^{-\lambda(\varphi - E(\varphi))} d\nu \leq (d_m)^{\frac{\lambda}{m}} (C_1)^{1-\frac{\lambda}{m}-\frac{\lambda\varepsilon}{\alpha}} (C_2)^{\frac{\lambda\varepsilon}{\alpha}}$.

$\forall \varphi \in \mathcal{H}_\omega$.

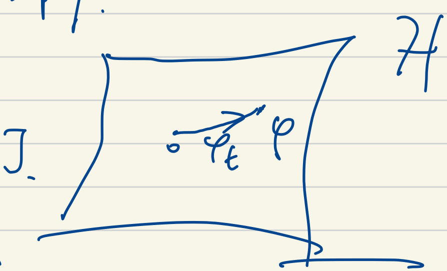
So we have $\delta_m^A(L) \geq \limsup_{m\to\infty} \delta_m^A$

$\square$.

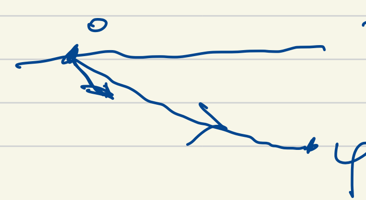
• Prop proof: $E(\varphi) \leq E_m(((1-\varepsilon)\varphi)_m) + \varepsilon \sup \varphi$.

Can assume $\sup \varphi = 0$.

Let φ_t be the "geodesic" from 0 to φ. $t \in [0,1]$.



Then φ_t is convex in t.



So $\underline{\dot{\varphi}_0 \leq 0}$.

$\varphi_t \in (\mathcal{H}, d_1)$

$\varphi_1 \in \mathcal{E}^1$.

Then look at $mL = \underbrace{mL - K_X}_{} + K_X$

$h^m \cdot e^{-m(1-\varepsilon)\varphi_t} \cdot \det\omega$ ← possibly singular Hermitian metric on $mL - K_X$.

"Berndtsson's convexity".

$$R_{h^m e^{-m(1-\varepsilon)\varphi_t}\det\omega} = Ric(\omega) + mR_h + m(1-\varepsilon)dd^c\varphi_t = \underbrace{Ric(\omega) + m\varepsilon\omega}_{\geq 0} + \underbrace{m\varepsilon\omega_{\varphi_t}}_{\geq 0}$$

Can choose h s.t. $R_h = \omega \in C(L)$.

So for m large enough, has ≥ 0 curvature.

Now, "Berndtsson's convexity" says that

☆ $E_m\left(((1-\varepsilon)\varphi_t)_m\right) = F(t)$ is convex in t.

Then $E_m(((1-\varepsilon)\varphi)_m) = F(1) = F(1) - F(0) \geq F'(0)$

$\underline{\underline{\text{e.x.}}}$ $\frac{(1-\varepsilon)}{d_m}\int_X \underbrace{\dot{\varphi}_0}_{\leq 0} \underbrace{\left(\sum_{i=1}^{d_m} |s_i|^2_{h^m}\right)\omega^n}_{\leq \frac{1}{(1-\varepsilon)}\frac{1}{V}}$ where $\{s_i\}$ is any H-ONB, $H = \int_X h^m(\cdot,\cdot)\omega^n$.

By Tian's Bergman kernel asymptotic we have 1990.

for $m \geq m_0(X, L, \omega, \varepsilon)$ s.t.

$$\frac{\sum_{i=1}^{d_m} |s_i|^2_{h^m}}{d_m} \leq \frac{1}{1-\varepsilon} \cdot \frac{1}{V}. \qquad V = \int_X \omega^n.$$

Then

$$E_m\left(\left((1-\varepsilon)\varphi\right)_m\right) \geq \frac{1}{V} \int_X \dot{\varphi}_0 \, \omega^n = \frac{d}{dt}\Big|_{t=0} E(\varphi_t).$$

E is linear along geodesics.

~~$E(\varphi_t)$ is linear in t~~ $\quad E(\varphi_1) = E(\varphi).$ $\square$.