Combinatorics, Lecture 6, 2022/5/26

§ Ramsey's Theorem

Def. An r-edge-coloring of K_n is a mapping $f: E(K_n) \to \{1, 2, \dots, r\}$. For 2-edge-coloring, we will often use the colors blue and red instead.

A sub-clique of K_n is monochromatic if all it edges have the same color.

Def For $k, \ell \geq 2$, the Ramsey number $R(k,\ell)$ is the smallest integer N such that any 2-edge-coloring of K_N has a blue K_k or a red K_ℓ.

Let us understand this definition a bit more:

$\Leftrightarrow$ $R(k,\ell) \leq L$ if and only if any 2-edge-coloring of K_L has a blue K_k or a red K_ℓ

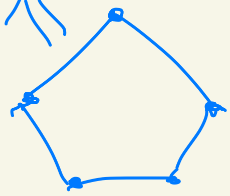
(2) $R(k,\ell) > M$ if and only if there exists some 2-edge-coloring of K_M, which has no blue K_k nor red K_ℓ.

Facts

(1) $R(k,\ell) = R(\ell,k)$ ✓

(2) $R(2,\ell) = \ell$, $R(k,2) = k$.

(3) $R(3,3) \le 6$ & $R(3,3) > 5$



$\Longrightarrow R(3,3) = 6$.

Ramsey's Thm. The Ramsey number $R(k,\ell)$ is finite. In fact, we have that

$$R(k,\ell) \le R(k-1,\ell) + R(k,\ell-1),$$

and thus in particular, $R(k,\ell) \le \binom{k+\ell-2}{k-1}$

pf. First, we prove $R(k,\ell) \le R(k-1,\ell) + R(k,\ell-1)$ (*)

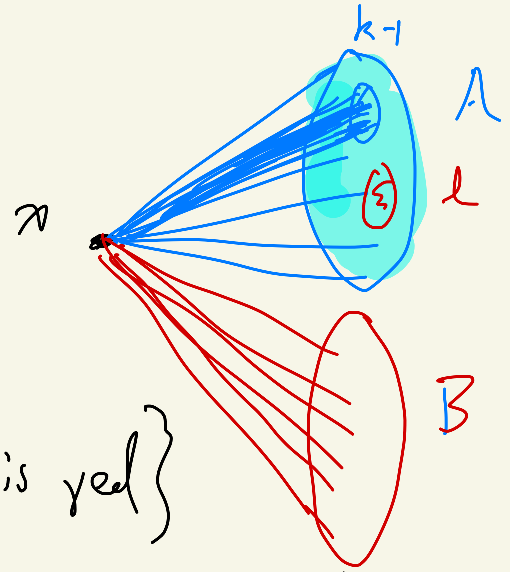
Let $n = R(k-1,\ell) + R(k,\ell-1)$.

So we want to show any 2-edge-coloring of K_n has a blue K_k or a red K_ℓ.

Consider a 2-edge-coloring of K_n.

Fix a vertex x in K_n.

Define $A = \left\{ \begin{array}{l} y \in V(K_n) \setminus \{x\} : \\ xy \text{ is blue} \end{array} \right\}$



and $B = \{ y \in V(K_n) \setminus \{x\} : xy \text{ is red} \}$

$\Rightarrow \quad |A| + |B| = n-1 = R(k-1, \ell) + R(k, \ell) - 1$

By Pigeonhole-principle, either $|A| \geq R(k-1, \ell)$

or $|B| \geq R(k, \ell-1)$.

Case 1 $\quad |A| \geq R(k-1, \ell)$.

Consider the complete graph induced on A, which has the restricted 2-edge-coloring.

By definition of $R(k-1, \ell)$, A contains a blue K_{k-1} or a red K_ℓ.

In the former case, we will have a blue K_k by adding the vertex x.

Case 2 $\quad |B| \geq R(k, \ell-1)$

The same proof.

This proves $(*)$.

Next, We will prove $R(k,l) \le \binom{k+l-2}{k-1}$ by induction on $k+l$. Base Case: $k=2$ or $l=2$ ✓

Assume this holds for any pair whose sum is less than $k+l$.

we have

$$R(k,l) \le R(k-1,l) + R(k,l-1)$$

$$\le \binom{k+l-3}{k-2} + \binom{k+l-3}{k-1} = \binom{k+l-2}{k-1}.$$

$\square$

Rmk. The diagonal Ramsey number

$$R(k,k) \le \binom{2k-2}{k-1} = \frac{(2k-2)!}{(k-1)!(k-1)!} \approx 2^{2k}$$

Thm 2. If $R(k-1,l)$ and $R(k,l-1)$ both are even, then we have

$$R(k,l) \le R(k-1,l) + R(k,l-1) - 1.$$

Pf: Let $n = R(k-1,l) + R(k,l-1) - 1$. Consider any 2-edge-coloring of K_n. n is $\boxed{\text{odd}}$

For $\forall x \in V(K_n)$, let $A_x = \{ y \in K_n - x : xy \text{ blue} \}$

$B_x = \{ \ldots : xy \text{ red} \}$.

The previous proof shows that

if $|A_x| \geq R(k-1, l)$ or $|B_x| \geq R(k, l-1)$,

then we are done.

So we may assume that for $\forall x$,

$$|A_x| \leq R(k-1, l) - 1 \text{ and } |B_x| \leq R(k, l-1) - 1$$

$$\Rightarrow n-1 = |A_x| + |B_x| \leq R(k-1, l) + R(k, l-1) - 2$$

$$= n-1$$

$$\Rightarrow \forall x, \quad |A_x| = \underbrace{R(k-1, l) - 1}_{\text{odd}} \ \& \ |B_x| = \underbrace{R(k, l-1) - 1}_{\text{odd}}$$

Consider the graph G consisting of all blue edges.

Then $n = |V(G)|$ is odd, and any vertex

has an odd degree. This is a contradiction

to the Handshaking lemma. $\boxtimes$

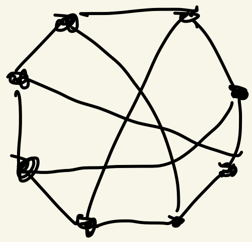
Corollary: $R(3,4) = 9$

pf: We know $R(2,4) = 4$ & $R(3,3) = 6$

both are even. By Thm 2,

$R(3,4) \le R(2,4) + R(3,3) - 1 = 9$.

To see $R(3,4) > 8$, we consider the following graph



No K_3 and no I_4 $\square$

§ Extensions of Ramsey's Theorem.

<u>Def</u>. For $k \geq 2$ and integers $s_1, s_2, \cdots, s_k \geq 2$, the Ramsey number $R_k(s_1, s_2, \cdots, s_k)$ is the least integer N such that any k-edge-coloring of K_N has a monochromatic K_{s_i} of the color i, for some $i \in [k]$.

(Multi-color Ramsey number)

<u>Exr.</u> $R_k(s_1, \cdots, s_k) < +\infty$

<u>Thm 3</u> (Schur's Thm) For any integer $k \geq 2$, there exists an integer $N = N(k)$ such that the

following holds. For any coloring $\varphi : [N] \to [k]$, there exist $x, y, z \in [N]$ such that

$$\varphi(x) = \varphi(y) = \varphi(z) \text{ and } x + y = z.$$

Pf: Let $N(k) = R_k(3, 3, \cdots, 3)$.

Fix $\varphi : [N] \to [k]$. We then define a k-edge-coloring of K_N as follows:

define the color of ij to be $\varphi(|i-j|)$.

By Ramsey number definition, we know there exists a monochromatic triangle ijl, where $i < j < l$.

$$\Longrightarrow \varphi(|i-j|) = \varphi(|i-l|) = \varphi(|j-l|)$$

Let $x = l-j$, $y = j-i$ and $z = l-i > 0$.

$$\Longrightarrow x + y = z \ \& \ \varphi(x) = \varphi(y) = \varphi(z). \ \boxtimes$$

Rmk: It also holds for $\varphi : [\geq N] \to [k]$.

Ex: show that x, y, z can be chosen to be distinct.

Next, we show the following restricted version of Fermat's last problem in $\mathbb{F}_q$.

Thm 4 (Schur) For any integer $m \geq 1$, there exists an integer $p(m)$ such that the following holds. For any prime $p \geq p(m)$, $x^m + y^m = z^m \pmod p$ has a solution in $\mathbb{F}_p \setminus \{0\}$.

Take $p(m) = N(m)$ be from Thm 3

pf: For prime p, consider the multiplicative group $\{1, 2, \ldots, p-1\} \cong \mathbb{F}_p^*$.

Let g be any generator of $\mathbb{F}_p^*$.

For $\forall x \in \mathbb{F}_p^*$, $\exists$ a unique (i, j) such that $x = g^{im+j} \pmod p$, where $1 \leq im + j \leq p-2$ and $0 \leq j \leq m-1$.

We define a coloring $\varphi : \mathbb{F}_p^* \to \{0, 1, \cdots, m-1\}$

by defining $\varphi(x) = j$.

By Thm 3, $p \geq p(m) \geq N(m)$

$\Rightarrow \exists\ x, y, z \in \mathbb{F}_p^*$ with

$\varphi(x) = \varphi(y) = \varphi(z) = j$ and $x + y = z$.

Let $x = g^{i_1 m + j} \bmod p$,

$y = g^{i_2 m + j} \bmod p$, and

$z = g^{i_3 m + j} \bmod p$.

$\Rightarrow g^{i_1 m + j} + g^{i_2 m + j} = g^{i_3 m + j} \bmod p$

$\Rightarrow g^{i_1 m} + g^{i_2 m} = g^{i_3 m} \bmod p$

Let $\alpha = g^{i_1}$, $\beta = g^{i_2}$, $\gamma = g^{i_3}$.

we have $\alpha^m + \beta^m = \gamma^m \pmod p$ $\boxed{\text{证}}$

§3 Hypergraph Ramsey number

Def. Let $r \geq 3$. An <u>r-uniform hypergraph</u> (i.e., <u>r-graph</u>) is a pair (V, E) such that $E \subseteq \binom{V}{r}$.

Let $K_n^{(r)}$ be the complete r-uniform hypergraph on n vertices, <u>i.e.</u> $K_n^{(r)} = (V, \binom{V}{r})$ with $|V| = n$.

Def. The <u>hypergraph Ramsey number</u> $R^{(r)}(s,t)$ is the least integer N such that any 2-edge-coloring of $K_N^{(r)}$ has a blue $K_s^{(r)}$ or a red $K_t^{(r)}$.

<u>Ex.</u> $\forall\, s,t \geq r$, $R^{(r)}(s,t) < +\infty$.

Erdős-Szekeres Thm For any integer n, there exists an integer $N(n)$ such that any collection of $N \geq N(n)$ points in the plane, no three on a line, has a subset of n points forming a convex n-gon.

pf. Using $R^{(3)}(n,n)$ $\boxed{}$

注意：从下周开始，

- 每周2晚上上3节课
- 每周4上午不再上课