Combinatorics, Lecture 4, 2022/5/19

§1. Sperner's Theorem

Def. Let $\mathcal{F} \subseteq 2^{[n]}$ be a family of subsets of $[n]$. We say $\mathcal{F}$ is an independent system (or independent), if for any two subsets $A, B \in \mathcal{F}$, we have $A \nsubseteq B$ and $B \nsubseteq A$.

$$\boxed{\text{Sperner's Theorem: For any independent system } \mathcal{F} \text{ of } [n], \text{ we have } |\mathcal{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}.}$$

Def. A chain is symmetric, if it consists of subsets of sizes $k, k+1, \cdots, n-k$ for some $k \geq 0$.

Thm 2. The family $2^{[n]}$ can be partitioned into a disjoint union of symmetric chains.

Pf 1: We use induction on n. $\boxed{\text{iZ}}$

Pf 2: For any $A \in 2^{[n]}$, we define a sequence " $a_1 a_2 \ldots a_n$ ", where

$$a_i = \begin{cases} \text{"("}, & \text{if } i \in A \\ \text{")"} & \text{if } i \notin A \end{cases}$$

We then define the "partial pairing of parentheses" as follows:

(1) First, we pair up all pires "( )" of adjacent parentheses,

(2) Then, we delete these already paired parentheses,

(3) Repeat the above process until nothing can be done.

Note that when this process stops, the remaining unpaired parentheses must look like the following:

$$))))))) (((((\quad (*)$$

We say two subsets $A, B \in 2^{[n]}$ have the same partial pairing, if the paired parentheses are the same (even in the same positions)

We can define an equivalence "$\sim$" on $2^{[n]}$ by letting $A \sim B$ iff A, B have the same partial pairing.

Claim: Every equivalent class in fact defines a symmetric chain. (练习)

Using this claim, $2^{[n]}$ can be partitioned into disjoint equivalent classes, which are disjoint symmetric chains. $\boxtimes$

<u>Second proof of Sperner's Thm</u>. Note that by definition, each symmetric chain contains exactly one subset in $\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$. This tells us that any partition of $2^{[n]}$ into symmetric chains has to consist of exactly $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ symmetric chains.

Consider any independent system $\mathcal{F}$ of $[n]$. Since $\mathcal{F}$ can contain at most one subset from each symmetric chain, we see

$$|\mathcal{F}| \leq \# \text{ sym. chains in } 2^{[n]} = \binom{n}{\lfloor \frac{n}{2} \rfloor}$$ $\boxtimes$

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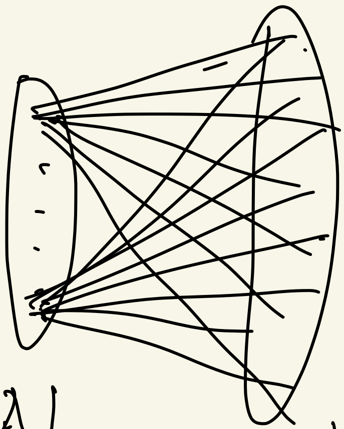
§2. Turán Type problem

Def. A graph G is <u>bipartite</u>, if $V(G)$ can be partitioned into two parts A and B such that each edge joins one vertex in A and another in B

A B

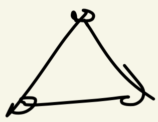
<u>e.g.</u> complete bipartite graph

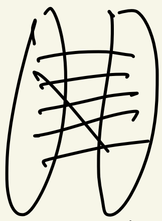
$K_{s,t}$ =



$s=|A|$ $|B|=t$

Def. Given a graph H, we say a graph G is <u>H-free</u>, if G doesn't contain H as its subgraph.

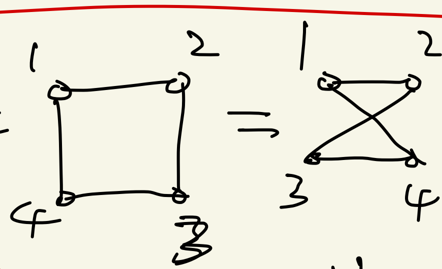
<u>e.g.</u> $H=$  $G=$ bipartite



A B

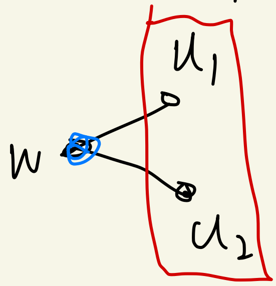
$\Rightarrow$ G is <u>H-free</u>

Def. The <u>Turán number</u> of H, denoted by $ex(n, H)$, is the maximum number of edges in an n-vertex H-free graph G.

$C_4 =$ 

$= K_{2,2}$

<u>**Thm 1**</u>. $ex(n, C_4) \le \frac{n}{4}\left(1+\sqrt{4n-3}\right)$.

<u>**Pf**</u>. We call 

a 2-path (a path of length 2).

Let G be an n-vtx C_4-free graph

We aim to show $e(G) \le \frac{n}{4}\left(1+\sqrt{4n-3}\right)$ ✓.

We will double count the total number

\# of all 2-paths in G.

① $\# = \sum_{w \in V(G)} \binom{d(w)}{2}$, where $d(w)$ is the degree of w in G.

2 For each fixed pair $\{u_1 u_2\}$,

there is at most one 2-path $u_1 w u_2$.

$$\Rightarrow \quad \# \leq \binom{n}{2}$$

Together, we have

$$\frac{n^2-n}{2} = \binom{n}{2} \geq \# = \sum_w \binom{d(w)}{2}$$

$$= \sum_w \frac{d(w)^2 - d(w)}{2}$$

$$= \frac{n}{2}\left(\frac{\sum_w d(w)}{n}\right)^2 - \frac{\sum_w d(w)}{2}$$

Cauchy-Schwarz inequality

$$\geq \frac{n}{2}\left(\frac{\sum_w d(w)}{n}\right)^2 - \frac{\sum_w d(w)}{2}$$

握手引理

$$= \frac{2\,e(G)^2}{n} - e(G)$$

$$\Rightarrow \quad \frac{2\,e(G)^2}{n} - e(G) \leq \frac{n^2-n}{2}$$

$$\Rightarrow \quad e(G) \leq \frac{n}{4}\left(1+\sqrt{4n-3}\right)$$ 证

Ex. Prove $ex(n, C_4) < \frac{n}{4}\left(1+\sqrt{4n-3}\right)$ (?)

Thm 2 (Kővári–Sós–Turán Theorem)

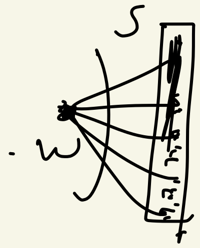
$$ex(n, K_{s,t}) \le \frac{1}{2}(t-1)^{1/s} \cdot n^{2-1/s} + \frac{1}{2}(s-1)n$$

for all $t, s \ge 2$.

Remark: The roles of s & t are symmetric

Pf: Let G be an n-vtx $K_{s,t}$-free graph with $\boxed{e(G) \ge \frac{1}{2} s n}$ (O.W, we are done)

We aim to show $e(G) \le \frac{1}{2}(t-1)^{1/s} \cdot n^{2-1/s} + \frac{1}{2}(s-1)n$.

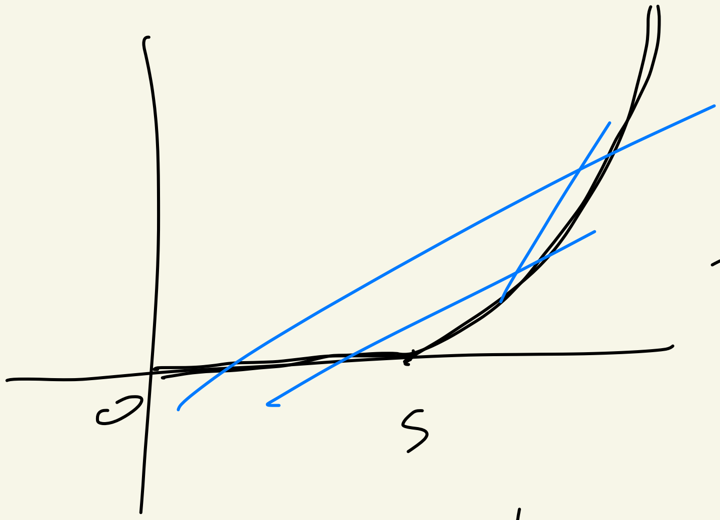
Count the number T of s-stars 

① $$T = \sum_{w \in V(G)} \binom{d(w)}{s}.$$

② $$T \le (t-1)\binom{n}{s}$$

Define $$f(x) = \begin{cases} 0, & x < s \\ \binom{x}{s} & x \ge s \end{cases}$$

$$= \frac{x(x-1)\cdots(x-s+1)}{s!}$$



$f(x)$ is convex

By Jensen's inequality,

$$\frac{(t-1)\binom{n}{s}}{n} \geq \frac{T}{n} = \frac{1}{n}\sum_w f(d(w))$$

$$\geq f\left(\frac{\sum_w d(w)}{n}\right) = f\left(\frac{2e(G)}{n}\right)$$

$d \leq \boxed{\frac{2e(G)}{n} \geq s}$

$$\geq \frac{(d-s+1)^s}{s!}$$

$d = \frac{2e(G)}{n}$

average degree

$$\Longrightarrow \frac{(d-s+1)^s}{s!} \leq \frac{(t-1)}{n}\binom{n}{s}$$

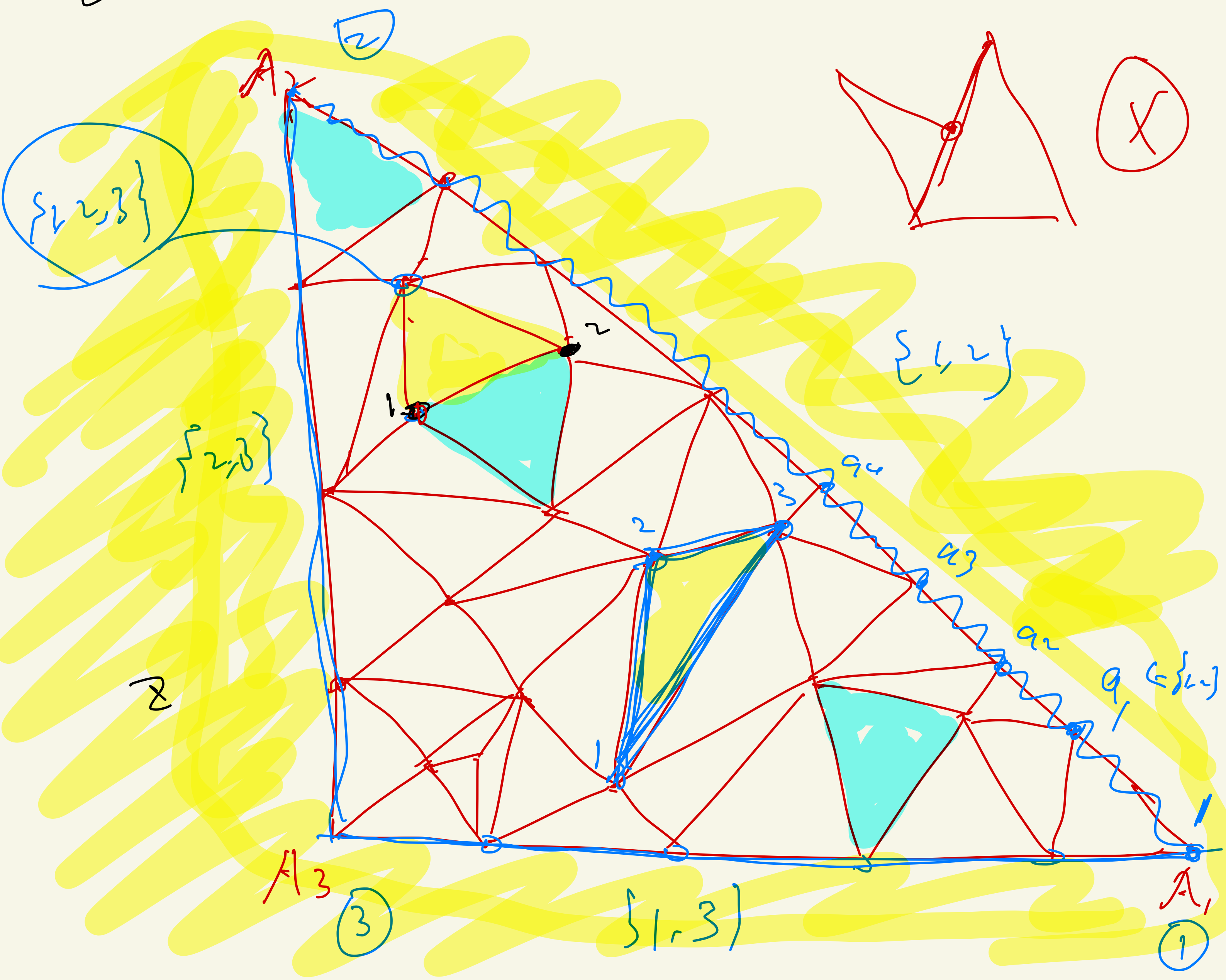
$$\Longrightarrow d \leq (t-1)^{\frac{1}{s}} n^{1-\frac{1}{s}} + (s-1). \quad \square$$

§3. Sperner's lemma

Recall a corollary:

In a graph G, if there exists a vertex of odd degree, then there are at least two vertices of odd degree.

Let us draw a triangle in the plane with 3 vertices.



Then we divide this triangle $\Delta = A_1A_2A_3$ into small triangles such that no triangle can have a vertex inside an edge of any other triangle.

Next, we assign 3 colors (say 1, 2, 3) to all vertices of these triangles, under the following rules:

(1) The vertex A_i is colored by $i \in [3]$.

(2) All vertices lying on the edge A_iA_j of the triangle $\Delta = A_1A_2A_3$ are assigned by the color i or j, for $\forall\, i, j \in [3]$.

(3) All interior vertices are colored by one of $\{1, 2, 3\}$

<u>Sperner's Lemma</u> For any assignment of colors described as above, there always exists a small triangle whose three vertices are assigned by three colors 1, 2, 3.

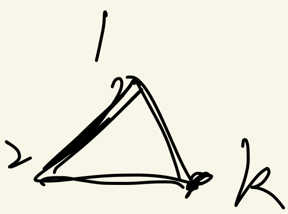
<u>Pf</u>: Define an <u>auxiliary graph</u> G as follows:

- Its vertices are the faces of small triangles and the outer face z.
- Two vertices of G are adjacent if the two corresponding faces are neighboring faces and the two endpoints of their common edge are colored by 1 and 2.

We consider the degree of any $v \in V(G)\setminus\{z\}$.

(1) If the face of v has NO two endpoints with colors 1 and 2, then $d_G(v)=0$.

(2) If the face of v has two endpoints with color 1 and 2, let k be the color of the third endpoint.



If $k \in \{1,2\}$, $d_G(v)=2$.

If $k=3$, $d_G(v)=1$ and this triangle is what we are looking for.

$\Rightarrow$ $d_{G}(v)$ is odd iff $d_{G}(v) = 1$

iff the face of v is what we want.

Now we claim that $d_{G}(z)$ must be odd

Consider the sequence of colors of the vertices in $A_1 A_2$, which is $\underset{\substack{\| \\ 1}}{a_0} \underbrace{a_1 \cdots a_k}_{\in \{1,2\}} \underset{\substack{\| \\ 2}}{a_{k+1}}$.

$\Rightarrow$ # $a_i a_{i+1}$ with different colors is odd !

By the corollary at the beginning of §3, we can find another vertex $v \in V(G) \setminus \{z\}$ with odd degree.

$\blacksquare$