2020-9-22 Kähler geometry

$z^i = x^i + \sqrt{-1}\, y^i \qquad i = 1, \dots, m$

$m = 1 \qquad z = x + \sqrt{-1}\, y$

$$TM \otimes \mathbb{C} \ni \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\,\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\,\frac{\partial}{\partial y}\right)$$

Let us define J as follows

$$J\frac{\partial}{\partial x} = \frac{\partial}{\partial y}, \qquad J\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}$$

$$J^2 = -\mathrm{id}$$

Then

$$J\frac{\partial}{\partial z} = \frac{1}{2} J\left(\frac{\partial}{\partial y} + \sqrt{-1}\,\frac{\partial}{\partial x}\right)$$

$$= \sqrt{-1}\,\frac{1}{2} J\left(\frac{\partial}{\partial x} - \sqrt{-1}\,\frac{\partial}{\partial y}\right)$$

$$= \sqrt{-1}\,\frac{\partial}{\partial z}$$

$$J\frac{\partial}{\partial \bar{z}} = -\sqrt{-1}\,\frac{\partial}{\partial \bar{z}}$$

Modelled on this example we write for (M, J)

$$T'M \overset{\text{def}}{=} \{ X \in TM \otimes \mathbb{C} \mid JX = \sqrt{-1}\, X \}$$

$$T''M \overset{\text{def}}{=} \{ X \in TM \otimes \mathbb{C} \mid JX = -\sqrt{-1}\, X \}$$

$$T''M = \overline{T'M}$$

For a complex manifold

$$T'M = \left\langle \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n} \right\rangle$$

holo tangent bundle

type (1,0) vector

$$T''M = \left\langle \frac{\partial}{\partial \bar{z}^1}, \dots, \frac{\partial}{\partial \bar{z}^n} \right\rangle$$

type (0,1) vector

$$C^\infty(T'M) \ni X = X^i \frac{\partial}{\partial z^i}, \quad X^i \in C^\infty$$

$$Y = Y^i \frac{\partial}{\partial z^i}$$

$$[X, Y] = XY - YX$$

$$= \left(X^j \frac{\partial Y^i}{\partial z^j} - Y^j \frac{\partial X^i}{\partial z^j} \right) \frac{\partial}{\partial z^i}$$

$$\in C^\infty(T'M)$$

$C^\infty(T'M)$ is involutive in the sense of

Frobenius. (involutive $\Leftrightarrow$ integrable)

<u>Prop</u> In order for an almost cmpl (M, J) comes from true complex manifold only if

(1) $[C^\infty(T'M), C^\infty(T'M)] \subset C^\infty(T'M)$.

<u>Theorem</u> (Newlander-Nirenberg)

Conversely if (1) is satisfied (M, J) comes from true complex manifold.

(See Donaldson-Kronheimer.)

If X_1, Y_1 are real vector fields and $X + \sqrt{-1}\,Y \in T'M$ then

$$J(X + \sqrt{-1}\,Y) = \sqrt{-1}\,(X + \sqrt{-1}\,Y)$$

$$JX = -Y \Leftrightarrow JY = X \qquad J^2 = -id$$

$$\therefore Y = -JX$$

So $T'M = \{X - \sqrt{-1}\,JX \mid X \in C^\infty(TM)\}$

Similarly $T''M = \{X + \sqrt{-1}\,JX \mid X \in C^\infty(TM)\}$

(1) $\Leftrightarrow [X - \sqrt{-1}JX, Y - \sqrt{-1}JY] \in T'M$

$\Leftrightarrow J[X - \sqrt{-1}JX, Y - \sqrt{-1}JY] = \sqrt{-1}[X - \sqrt{-1}JX, Y - \sqrt{-1}JY]$

$\Leftrightarrow J[X,Y] - J[JX, JY] = [X, JY] + [JX, Y]$

$- J[JX, Y] - J[X, JY] = [X,Y] - [JX, JY]$

$N(X,Y) = [X,Y] - [JX, JY] + J[JX, Y] + J[X, JY]$

N is called the Nijenhuis tensor

$$\left(\begin{array}{l} N(fX, gY) = fg\,N(X,Y) \quad f, g \in C^\infty(M) \\ X_p = X'_p \Rightarrow N(X_p, Y) = N(X'_p, Y) \end{array} \right.$$

(1) $\Leftrightarrow N \equiv 0$

Theorem (Newlander–Hirenberg)

(M, J) is a complex mfd $\Leftrightarrow N \equiv 0$.

<u>Def</u> J is integrable $\overset{\text{def}}{\Leftrightarrow} N = 0$

We consider <u>integrable J</u>. (an)

<u>Def</u> Riemannian metric g is called a <u>Hermitian metric</u> if g is J-invariant

i.e. $g(JX, JY) = g(X, Y)$ for all X, Y.

Def $\omega(X,Y) = g(JX,Y)$

Lemma ω is a 2-form, i.e. ω is skew sym.

$\because$ $\omega(Y,X) = g(JY,X) = g(\underbrace{JJY}_{-Y}, JX)$

g sym

$= -g(JX,Y) = -\omega(Y,X)$ $\because$

Def g is a Kähler metric $\overset{\text{def}}{\Longleftrightarrow}$ $d\omega = 0$.

Def (M, g, J) with $d\omega = 0$ is called a <u>Kähler manifold</u>.

ω is called the Kähler form.

$[\omega] \in H^2_{DR}(M)$ is called the Kähler class

Def Sometimes, a complex manifold admitting a Kähler metric is called a Kähler manifold.

Def M is non-Kähler $\overset{\text{def}}{\Longleftrightarrow}$ M does not admit a Kähler metric

Ex. If $H^2_{DR}(M, \mathbb{R}) = 0$ then M is non-Kähler.

$\boxed{\text{Extend } g, D, R \text{ and etc in the } \mathbb{C}\text{-linear way.}}$

$z^1, \dots, z^n$ local holo coordinates.

$$\bar{z}^1, \dots, \bar{z}^n \longrightarrow z^{\bar{1}}, \dots, z^{\bar{n}}$$

$$\{A, B, C, \dots\} = \{1, \dots, n, \bar{1}, \dots, \bar{n}\}$$

$$\frac{\partial}{\partial \bar{z}^i} = \frac{\partial}{\partial z^{\bar{i}}}, \quad d\bar{z}^i = dz^{\bar{i}}$$

$$g_{AB} = g\left(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B}\right)$$

<u>Lemma</u> $g_{ij} = 0$, $g_{\bar{i}\bar{j}} = 0$

$\because$ $$g_{ij} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = g\left(J\frac{\partial}{\partial z^i}, J\frac{\partial}{\partial z^j}\right)$$

$$= g\left(\sqrt{-1}\frac{\partial}{\partial z^i}, \sqrt{-1}\frac{\partial}{\partial z^j}\right) = -g_{ij}$$

$$\therefore g_{ij} = 0$$

$\because$

So, we only have $g_{i\bar{j}} = g_{\bar{j}i}$

<u>Lemma</u> $\omega = \sqrt{-1}\, g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$.

$g_{\bar{i}j}\, d\bar{z}^i \wedge dz^j$

$\because$

$$\omega\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right) = g\left(J\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right)$$

$$= g\left(\sqrt{-1}\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right) = \sqrt{-1}\,\underline{g_{ij}} = 0$$

$$\omega\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial \bar{z}^j}\right) = 0.$$

$$\omega\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right) = g\left(J\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right)$$

$$= \sqrt{-1}\, g_{i\bar{j}}$$

$$\therefore \omega = \sqrt{-1}\, g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$$

$\because$

$$\nabla_{\frac{\partial}{\partial z^A}} \frac{\partial}{\partial z^B} = \Gamma^C_{AB} \frac{\partial}{\partial z^C}$$

Γ^C_{AB} is called the Christoffel symbol.

Torsion-free

$$\Longleftrightarrow \nabla_X Y - \nabla_Y X = [X, Y]$$

$$\nabla_{\frac{\partial}{\partial z^A}} \frac{\partial}{\partial z^B} - \nabla_{\frac{\partial}{\partial z^B}} \frac{\partial}{\partial z^A} = \left[\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B}\right] = 0.$$

$$\therefore \Gamma^C_{AB} = \Gamma^C_{BA} \quad \text{"torsion-free symmetric"}$$

As we extended $\mathbb{C}$-linearly

$$\overline{\Gamma^C_{AB}} = \Gamma^{\bar{C}}_{\bar{A}\bar{B}}.$$

<u>Prop</u> Let (M, J) be a complex manifold and g a Hermitian metric. Then the following are equivalent:

(i) $d\omega = 0$, i.e. g is Kähler

(ii) $\nabla J = 0$

(iii) $\Gamma^C_{AB} = 0$ possibly except for Γ^k_{ij} and $\Gamma^{\bar{k}}_{\bar{i}\bar{j}}$.

<u>Proof</u> (ii) $\Rightarrow$ (i) $\nabla_X(JY) = (\nabla_X J)Y + J\nabla_X Y$

$$d\omega(X,Y,Z) = X\omega(Y,Z) + Y\omega(Z,X) + Z\omega(X,Y)$$
$$-\omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y)$$
$$= X g(JY,Z) + Y g(JZ,X) + Z g(JX,Y)$$
$$- g(J[X,Y],Z) - g(J[Y,Z],X) - g(J[Z,X],Y)$$

$$= g((\nabla_X J)Y, Z) + g(J\nabla_X Y, Z) + g(JY, \nabla_X Z) \quad -g(Y, J\nabla_X Z)$$

$$+ g((\nabla_Y J)Z, X) + g(J\nabla_Y Z, X) + g(JZ, \nabla_Y X)$$

$$+ g((\nabla_Z J)X, Y) + g(J\nabla_Z X, Y) + \underline{g(JX, \nabla_Z Y)} \quad -g(X, J\nabla_Z Y)$$

$$- g(J[X,Y], Z) - g(J[Y,Z], X) - g(J[Z,X], Y)$$

$$= g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y)$$

$$\therefore \nabla J = 0 \implies d\omega = 0$$

$$(\nabla_X J)(JY) = \nabla_X(J^2 Y) - J\nabla_X JY$$

(i) $\Rightarrow$ (ii)

$$0 = d\omega(X, Y, Z) - d\omega(X, JY, JZ)$$

$$= g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y)$$

$$- g((\nabla_X J)JY, JZ) - g((\nabla_{JY} J)JZ, X) - g((\nabla_{JZ} J)X, JY)$$

$$= g(\nabla_X(JY) - J\nabla_X Y, Z) + g(\nabla_Y(JZ) - J\nabla_Y Z, X)$$

$$+ g(\nabla_Z(JX) - J\nabla_Z X, Y) + g(\nabla_X Y - J\nabla_X(JY), JZ)$$

$$- g(\nabla_{JY}(-Z) - J\nabla_{JY}(JZ), X) - g(\nabla_{JZ}(JX) - J\nabla_{JZ} X, JY)$$

$$= 2g(\nabla_X Y, JZ) + 2g(\nabla_X (JY), Z)$$

$$+ g(\nabla_Z (JX), Y)_{\Delta} + g(\nabla_Z X, JY)_{\square}$$

$$+ g(J\nabla_Y (JZ), JX)_{4} + g(\nabla_Y Z, JX)_{1}$$

$$+ g(J\nabla_{JY} Z, JX)_{3} - g(\nabla_{JY} JZ, JX)_{2}$$

$$- JZ\, g(X, Y)_{5} + g(JX, \nabla_{JZ} JY)_{2}$$

$$+ JZ\, g(X, Y)_{5} - g(JX, J\nabla_{JZ} Y)_{4}$$

$$\begin{array}{l} \cancel{2g(JX, Y)} \\ \hline -g(JX, \nabla_Z Y)_{1} \\ \cancel{+2g(X, JY)} \\ -g(JX, J\nabla_Z JY)_{3} \end{array}$$

$$= 2g((\nabla_X J)Y, Z) + \Big(g(JX, [Y, Z]_{1} - [JY, JZ]_{2} + J[JY, Z]_{3} + J[Y, JZ]_{4} \Big)$$

$$= 2g((\nabla_X J)Y, Z) + g(JX, \underbrace{N(Y, Z)}_{=0})$$

J integrable

$$= 2g((\nabla_X J)Y, Z)$$

$\therefore d\omega = 0 \Rightarrow \nabla J = 0.$